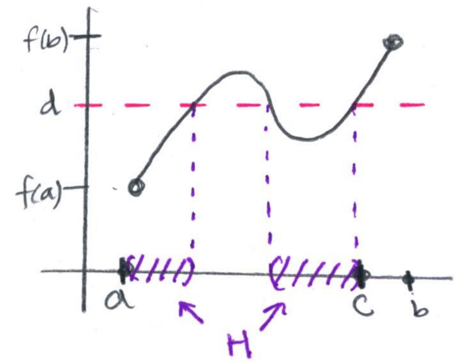
Recap from last time

Intermediate Value Thm: Let f be continuous on $[a,b]$, and $f(a) < f(b)$ given d with $f(a) < d < f(b)$. There exists c with $a < c < b$ and $f(c) = d$



proof:

let $H = \{x \mid a < x < b \text{ and } f(x) < d\}$

We showed (1) $H \neq \emptyset$

(2) b is an upper bound for H

so, $\sup(H)$ exists. Let $c = \sup(H)$

Note: Last thing we showed was $c < b$

Note that $a < c$

Why? since $H \neq \emptyset$, $\exists\, h \in H$

since $h \in H$, we have $a < h$

since $c = \sup(H)$, we have $h \leq c$ so, $a < h \leq c$.

We now need to show that $f(c) = d$

We do this by showing that $f(c) > d$ and $f(c) < d$ cannot happen

- Suppose $f(c) < d$

Then $d - f(c) > 0$. Let $\varepsilon = d - f(c)$

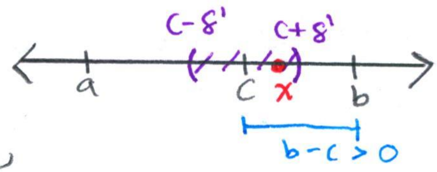
since $a < c < b$, we know that f is continuous at c.

so there exists $\delta' > 0$, where if $x \in [a,b]$ and $|x - c| < \delta'$

then $|f(x) - f(c)| < \varepsilon = d - f(c)$

* Always true $y \leq |y|$

Let us assume $\delta' < b - c$



so if $x \in [a,b]$ and $|x - c| < \delta'$ then,

$f(x) - f(c) \leq |f(x) - f(c)| < d - f(c)$

so for $x \in [a,b]$ with $|x - c| < \delta'$ we have $f(x) < d$

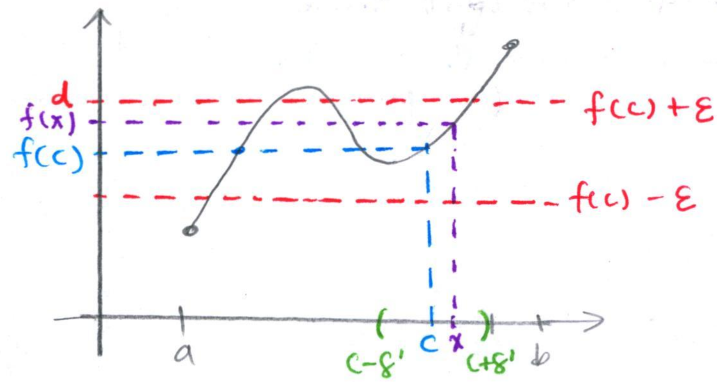
that is $x \in H$

For example if $x = c + \frac{1}{2}\delta'$,

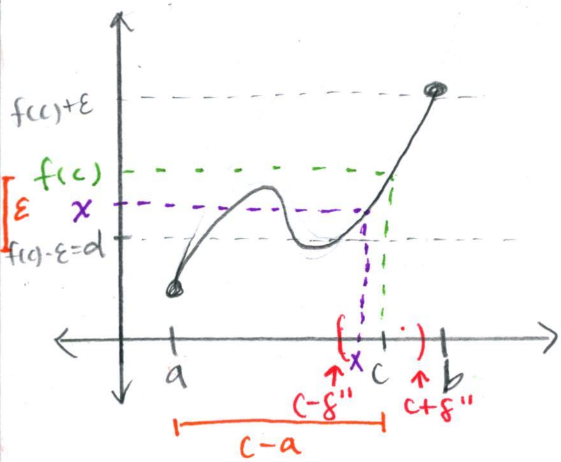
then $c < x < b$ and $f(x) < d$

then $x \in H$ and contradicts

the fact that $c = \sup(H)$.



- Suppose $f(c) > d$



Let $\varepsilon = f(c) - d > 0$

since f is continuous at c $\exists$ $\delta'' > 0$

where if $x \in [a,b]$ and $|x - c| < \delta''$ then

$|f(x) - f(c)| < \varepsilon = f(c) - d$

- we may assume $\delta'' < c - a$

so if $x \in [a,b]$ and $|x-c| < \delta''$ then $f(c) - f(x) \leq |f(x) - f(c)| < f(c) - d$

That is, if $x \in [a,b]$ and $|x - c| < \delta''$, then $d < f(x)$.

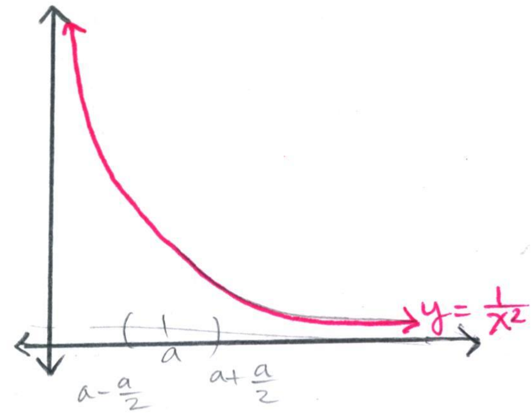
so, $(c-\delta'', c+\delta'') \cap H = \emptyset$.

However, by the useful sup fact, there must exists

$h \in H$ with $c - \delta'' < h < c$.

contradiction! ☒

Homework 4 #4



Let $f(x) = \frac{1}{x^2}$, let $a \in \mathbb{R}$, $a \neq 0$.

Show that f is continuous at a.

Proof: Let's assume $a > 0$

(A similar proof will work for $a < 0$)

Let $\varepsilon > 0$,

we need to find $\delta > 0$ where if $x \neq 0$ (ie. x is in the domain of f) and $|x-a| < \delta$ then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$.

Note that,

control via δ

need to bound by a number

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{a^2 - x^2}{x^2 a^2}\right| = \frac{|a-x||a+x|}{|x|^2\,|a|^2} = \frac{\boxed{|x-a|}\,\boxed{|x+a|}}{\boxed{|x|^2\,|a|^2}}$$

Suppose $\delta \leq \frac{a}{2}$

Suppose $|x-a| < \delta \leq \frac{a}{2}$ so, $a - \frac{a}{2} < x < a + \frac{a}{2}$. That is, $\frac{a}{2} < x < \frac{3a}{2}$.

So, $\frac{3a}{2} < x < \frac{5a}{2}$ Thus, $|x+a| < \frac{5a}{2}$.

Also, $\left(\frac{a}{2}\right)^2 < x^2 < \left(\frac{3a}{2}\right)^2$. So, $|x^2| > \frac{a^2}{4}$. Thus $\frac{1}{|x^2|} < \frac{4}{a^2}$

Thus if $|x-a| < \frac{a}{2}$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x-a||x+a|}{|x^2|\,|a^2|} < \left(|x-a|\right)\left(\frac{5a}{2}\right)\left(\frac{4}{a^2}\right)\left(\frac{1}{a^2}\right)$$

So if $|x-a| < \frac{a}{2}$, we have $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < |x-a| \cdot \frac{10}{a^3}$

Let $\delta = \min\left\{\frac{a}{2}, \frac{\varepsilon}{\left(\frac{10}{a^3}\right)}\right\}$, Therefore, if $|x-a| < \delta$,

then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \underset{\uparrow}{<} |x-a| \frac{10}{a^3} \underset{\uparrow}{<} \frac{\varepsilon}{\left(\frac{10}{a^3}\right)} \cdot \left(\frac{10}{a^3}\right) = \varepsilon$ ☒

$|x-a| < \frac{a}{2}$ $\quad$ $|x-a| < \frac{\varepsilon}{\left(\frac{10}{a^3}\right)}$

HW 4 #7 Let $f: D \to \mathbb{R}$ be continuous on D where $D \subseteq \mathbb{R}$

(a) suppose $\lim_{n\to\infty} a_n = L$ and $L \in D$ and $a_n \in D \ \forall n$.

Then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$

Ex:

$$\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}} = e^0 = 1$$

$\uparrow$ $f(x) = e^x$ is continuous $\forall x$.

new sequence $f(a_1), f(a_2), f(a_3), \ldots \to f(L)$

Proof: Let $\varepsilon > 0$

goal: we want to find $N > 0$ where if $n \geq N$ then $|f(a_n) - f(L)| < \varepsilon$

since $L \in D$ and f is continuous on D, f is continuous at L. Thus there exists $\delta > 0$ where if $|x - L| < \delta$ then $|f(x) - f(L)| < \varepsilon$

since $\lim_{n\to\infty} a_n = L$, $\exists\ N > 0$ where if $n \geq N$ then $|a_n - L| < \delta$

so if $n \geq N$ then $|a_n - L| < \delta$ and so $|f(a_n) - f(L)| < \varepsilon$ $\boxtimes$

HW 3 #1 (d) $\lim_{x\to\infty} \frac{2x}{x^2+1}$ (Modified) $\lim_{x\to\infty} \frac{2x^{10}}{x^{100} + 7x^4}$

proof: Let $\varepsilon > 0$

we have that $\left|\frac{2x^{10}}{x^{100}+7x^4} - 0\right| = \left|\frac{2x^{10}}{x^{100}+7x^4}\right| \underset{\uparrow}{=} \frac{2x^{10}}{x^{100}+7x^4} = \frac{2x^6}{x^{96}+7}$

assume $x > 0$ since $x \to \infty$

$$\frac{2x^6}{x^{96}+7} < \frac{2x^6}{x^{96}} = \frac{2}{x^{90}}$$

and $\frac{2}{x^{90}} < \varepsilon$ iff $\frac{2}{\varepsilon} < x^{90}$ iff $\sqrt[90]{\frac{2}{\varepsilon}} < x$

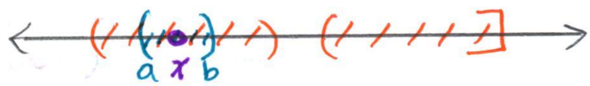
Let $N > \sqrt[90]{\frac{2}{\varepsilon}}$. If $x \geq N > \sqrt[90]{\frac{2}{\varepsilon}}$, then $\left|\frac{2x^{10}}{x^{100}+7x^4} - 0\right| < \varepsilon$ $\boxtimes$

Open and Closed Subsets of $\mathbb{R}$ (Topic 5)

Def: Let $S \subseteq \mathbb{R}$

We say that $x \in \mathbb{R}$ is an **interior point of S** if there exists $a, b \in \mathbb{R}$, $a < b$ with $x \in (a,b) \subseteq S$

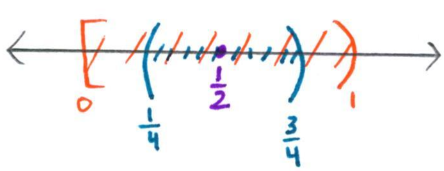
▨ - S
▨ - (a,b)



x sits inside an interval (a,b) that's completely contained in S.

Example: $S = [0,1)$

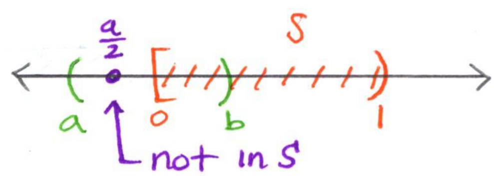
is $x = \frac{1}{2}$ an interior point of S?



yes, $\frac{1}{2} \in \left(\frac{1}{4}, \frac{3}{4}\right) \subseteq S$

is $x = 0$ an interior point of S?

NO, Any interval (a,b) with $0 \in (a,b)$ will not be contained in S



Ex: $-\frac{a}{2} \notin S$

but $-\frac{a}{2} \in (a,b)$

Def: Let $S \subseteq \mathbb{R}$

S is called **open** if every $x \in S$ is an interior point of S

Ex: $S = [0,1)$

is not open since $0 \in S$, but 0 is not an interior point of S.

Fact: If $a, b \in \mathbb{R}$ and $a < b$

then $S = (a,b)$ is an open set

proof: let $x \in S$

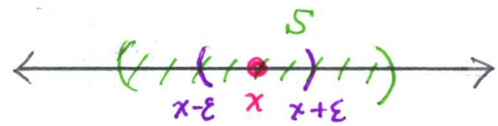
we need to show that x is an interior point of $S = (a, b)$

Since $x \in (a, b) \subseteq S$, x is an interior point of S ☑

Proposition: Let $S \subseteq \mathbb{R}$ (This is another way to say that x is an interior pt. of S)

Then S is open iff for every $x \in S$ there exists $\varepsilon > 0$

so that $(x-\varepsilon, x+\varepsilon) \subseteq S$



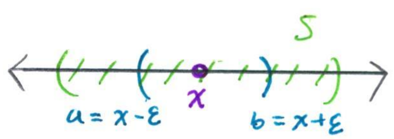
This prop. will follow from the following.

Proposition: Let $S \subseteq \mathbb{R}$

$x \in \mathbb{R}$ is an interior point of S iff $\exists\, \varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq S$

proof:

($\Leftarrow$) Let $x \in S$ such that $\exists\, \varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq S$

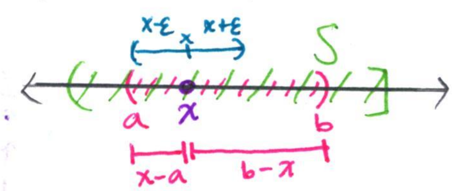


Let $a = x - \varepsilon$, $b = x + \varepsilon$, then $x \in (a, b) \subseteq S$

so x is an interior point.

($\Rightarrow$) Let $x \in S$ be an interior point of S. Then $\exists\, a, b \in \mathbb{R}$,

$a < b$, with $x \in (a, b) \subseteq S$.



Let $\varepsilon = \min\{x-a, b-x\}$

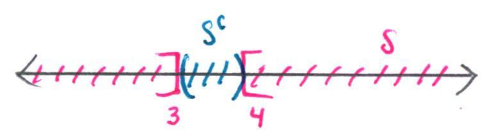
Then, $(x-\varepsilon, x+\varepsilon) \subseteq (a, b) \subseteq S$ ☑

Def: let $S \subseteq \mathbb{R}$

we say that S is **closed** if $S^c = \mathbb{R} \setminus S = \mathbb{R} - S = \{x \in \mathbb{R} \mid x \notin S\}$ is open.

(complement of S ; all real #s NOT in S)

Example: $S = (-\infty, 3] \cup [4, \infty)$



$S^c = \mathbb{R} \setminus S = (3, 4)$ is open

So, S is closed.