Last time:

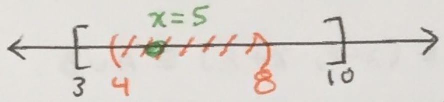
- $S \subseteq \mathbb{R}$

x is an interior poin of S if

(1) There exists $a, b \in \mathbb{R}$, $a < b$ with

$x \in (a, b) \subseteq S$

Ex: (1) $S = [3, 10]$

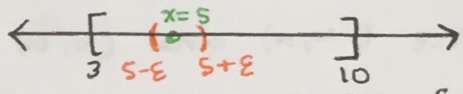


5 is an interior point of S because $5 \in (4, 8) \subseteq S$

(2) There exists $\varepsilon > 0$ with

$x \in (x - \varepsilon, x + \varepsilon) \subseteq S$

Ex: (2) $S = [3, 10]$ $\varepsilon = 0.5$



5 is an interior point of S b/c $5 \in (5 - 0.5, 5 + 0.5) = (4.5, 5.5) \subseteq S$

- $S \subseteq \mathbb{R}$ is open if every $x \in S$ is an interior point of S.

Ex: Last time we proved that sets of the form $S = (a, b)$ is open $S = (-2, 10)$ is open

- $S \subseteq \mathbb{R}$ is closed if $S^c = \mathbb{R} \setminus S$ is open

HW 5

(a) – $\mathbb{R}$ is open

(b) – $\emptyset$ is open ($\emptyset = \{\ \}$ is the empty set)

(c) – If $a \in \mathbb{R}$ then $(-\infty, a)$ is open and (a, ∞) is open.

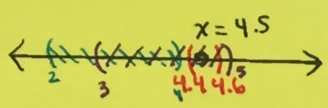
(d) – If $A, B \subseteq \mathbb{R}$ are both open then $A \cup B$ is open.

proof of (d):

Suppose A and B are open subsets of $\mathbb{R}$

We want to show that $A \cup B$ is open which can be done by showing that every $x \in A \cup B$ is an interior point of $A \cup B$.

Ex: $A = (3, 5)$ $B = (2, 4)$



$A \cup B = (2, 5)$

Let $x \in A \cup B$, so $x \in A$ or $x \in B$

Case 1: $x \in A$

since A is open, x is an interior pt. of A

therefore $\exists\ a, b \in \mathbb{R}$, $a < b$ where $x \in (a, b) \subseteq A$

so $x \in (a, b) \subseteq A \subseteq A \cup B$. Thus x is an interior pt. of $A \cup B$.

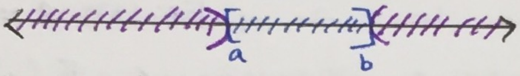
case 2: $x \in B$

since B is open, x is an interior point of B. So there exists $\varepsilon > 0$ where $x \in (x-\varepsilon, x+\varepsilon) \subseteq B$, since $B \subseteq A \cup B$,

$x \in (x-\varepsilon, x+\varepsilon) \subseteq A \cup B$, so x is an interior pt. of $A \cup B$ $\boxtimes$

Ex: Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a,b]$ is closed.

Proof: We have that $\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$

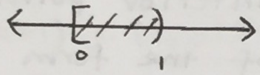


since $(-\infty, a)$ and (b, ∞) are both open, $(-\infty, a) \cup (b, \infty)$ is open

so $[a,b]$ is closed $\boxtimes$

Q? Can we have a set that is not open and not closed?

Ex: $[0, 1)$



- S is Not open because $0 \in S$ and 0 is not an interior point of S
- Is S closed?

$\mathbb{R} \setminus S = (-\infty, 0) \cup [1, \infty)$ is not open

because $1 \in \mathbb{R} \setminus S$ and is not an interior pt. of $\mathbb{R} \setminus S$

so S is Not closed.

Are these sets open or closed?

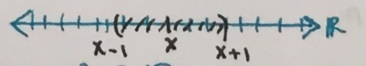
Closed	Open	Example sets
NO	NO	$[0,1)$
YES	NO	$[0,1]$
NO	YES	$(0,1)$
YES	YES	$\mathbb{R}$, $\varnothing$

$\mathbb{R} \setminus (0,1) = (-\infty, 0] \cup [1, \infty)$ is not open since 0 is in $\mathbb{R} \setminus (0,1)$ but not an interior point.

So, $(0,1)$ is not closed.

$\mathbb{R}$ is open

let $x \in \mathbb{R}$



Then $x \in (x-1, x+1) \subseteq \mathbb{R}$ $\boxtimes$

Theorem: Let $A \subseteq \mathbb{R}$ then A is open iff

A is a countable disjoint union of open intervals of $\mathbb{R}$

That is, if A is open iff

$$A = \bigcup_n (a_n, b_n)$$ ← The union could be infinitely countable or finite. or empty

where $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ if $i \neq j$

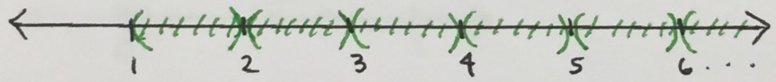
Proof: Handout from "Elements of Real analysis"

Gaskill\ Narayanaswami p. 143-144 ☒

Ex: $A = (9, 23) \cup (39, 41) \cup (47, 10{,}000{,}000)$

A is open since the union of 3 disjoint intervals.

Ex: $B = \bigcup_{n=1}^{\infty} (n, n+1) = (1,2) \cup (2,3) \cup (3,4) \cup (4,5) \cup \ldots$

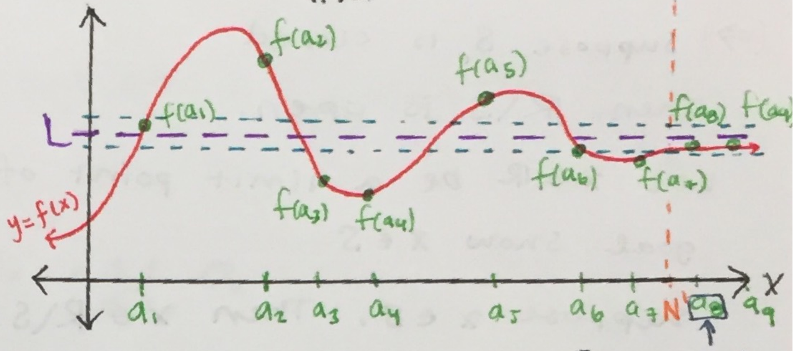


B is open. it's the union of a countable number of open intervals.

P.1 10/31 Wednesday week 11

HW 3

#4(a) Suppose that $\lim_{x\to\infty} f(x) = L$ and (a_n) is an unbounded increasing sequence of real numbers. Then $\lim_{n\to\infty} f(a_n) = L$



Proof: Let $\varepsilon > 0$,

since $\lim_{x\to\infty} f(x) = L$, there exists $N' > 0$ where if $x \geq N'$ then $|f(x) - L| < \varepsilon$,

Since (a_n) is unbounded and increasing then there exists $N > 0$ where if $n \geq N$ then $N' \leq a_n$

-In this picture if $n \geq 8$ then $N' \leq a_n$

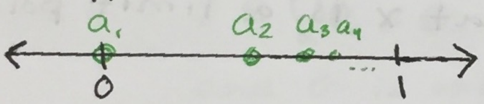
So if $n \geq N$ then $N' \leq a_n$ and so $|f(a_n) - L| < \varepsilon$ ☒

Need both unbounded and increasing for a_n.

Ex: $a_n = 1 - \frac{1}{n}$

- increasing
- bounded

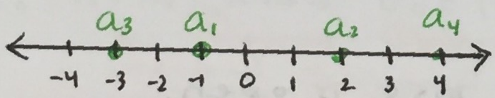
← If $N' > 1$ can't do proof



Ex: $a_n = (-1)^n n$

- unbounded
- not increasing

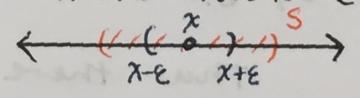
← can't ever find N with $a_n \geq N'$ $\forall n \geq N$ because a_n alternates.



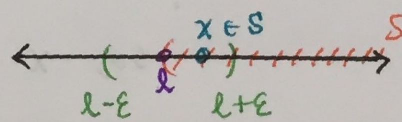
Back to Open and Closed Sets

Recall Let $S \subseteq \mathbb{R}$

- S is **open** if for every $x \in S$ there exists $\varepsilon > 0$ where $(x-\varepsilon, x+\varepsilon) \subseteq S$ } every $x \in S$ is an interior point of S.



- S is **closed** if $S^c = \mathbb{R} \setminus S$ is open
- ℓ is a **limit point** of S if for every $\varepsilon > 0$ there exists $x \in S$ with $\underbrace{0 < |x - \ell| < \varepsilon}$



means:

(1) $x \neq \ell$

(2) $\ell - \varepsilon < x < \ell + \varepsilon$

Theorem: Let $S \subseteq \mathbb{R}$

S is closed iff S contains all it's limit points.

Proof:

($\Rightarrow$) Suppose S is closed

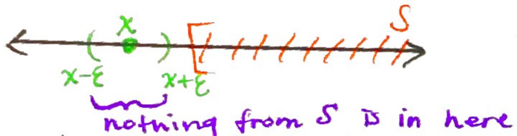
Then $\mathbb{R} \setminus S$ is open.

Let $x \in \mathbb{R}$ be a limit point of S.

goal: show $x \in S$

Suppose $x \in S$, Then $x \in \mathbb{R} \setminus S$.

Since $\mathbb{R} \setminus S$ is open, x is an interior point of $\mathbb{R} \setminus S$



So there exists $\varepsilon > 0$ where $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R} \setminus S$

That is, $S \cap (x-\varepsilon, x+\varepsilon) = \emptyset$.

This contradicts the fact that x is a limit point of S

therefore, $x \notin S$ can't be so.

Thus, $x \in S$. So, S contains all its limit points.

($\Leftarrow$) Suppose S contains all it's limit points.

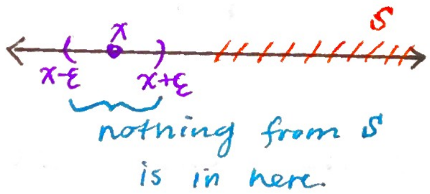
goal: Show S is closed.

We do this by showing that $\mathbb{R} \setminus S$ is open.

Let $x \in \mathbb{R} \setminus S$. We need to show x is an interior point of $\mathbb{R} \setminus S$.

Since S contains all it's limit points and $x \notin S$

x is not a limit point of S.



Thus there exists $\varepsilon > 0$

where $S \cap (x-\varepsilon, x+\varepsilon) = \emptyset$

That is, $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R} \setminus S$

So, x is an interior point of $\mathbb{R} \setminus S$

Thus, $\mathbb{R} \setminus S$ is open, so S is closed

$\boxtimes$

Compactness

Def: Let $S \subseteq \mathbb{R}$

An **open cover** of S is a collection $X = \{\mathcal{O}_\alpha\}$ of open sets such that $S \subseteq \bigcup_\alpha \mathcal{O}_\alpha$

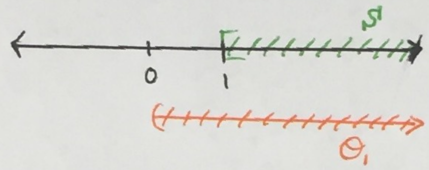
If $X' \subseteq X$ such that $S \subseteq \bigcup_{\substack{\alpha \\ \mathcal{O}_\alpha \in X'}} \mathcal{O}_\alpha = \bigcup_{\mathcal{O}_\alpha \in X'} \mathcal{O}_\alpha$

then X' is a **subcover** of X. If X' is as above and X' is finite, then X' is called a **finite subcover**.

Ex: $S = [1, \infty)$

let $X_1 = \{(0, \infty)\}$

$\uparrow$
$\mathcal{O}_1$

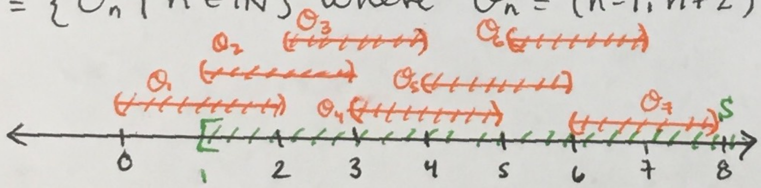


$S \subseteq \bigcup_\alpha \mathcal{O}_\alpha = \mathcal{O}_1 = (0, \infty)$

and $\mathcal{O}_1$ is open, so, X_1 is an open cover of S.

Ex: $S = [1, \infty)$

let $X_2 = \{\mathcal{O}_n \mid n \in \mathbb{N}\}$ where $\mathcal{O}_n = (n-1, n+2)$ ← all open



Then $S \subseteq (0, \infty) = \bigcup_{n=1}^{\infty} \mathcal{O}_n$, so X_2 is an open cover of S.