Compactness: $S \subseteq \mathbb{R}$ is compact if every open cover of S contains a finite subcover.

Heine-Borel Theorem:

Let $K \subseteq \mathbb{R}$

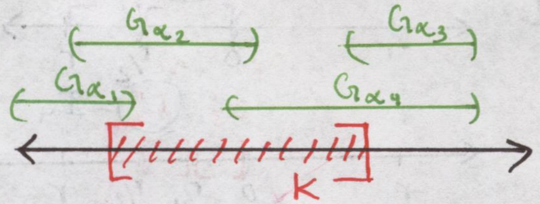
K is compact iff K is closed and bounded

proof:

($\Rightarrow$) See Handout.

($\Leftarrow$) Suppose K is closed and bounded

Let $G = \{G_\alpha\}$ be an open cover of K. That is, each G_α is open and $K \subseteq \bigcup_\alpha G_\alpha$. We want to show that K is contained in some finite subcover from G.



note: this picture only has a finite # of G_α but there are infinitely many in G in proof.

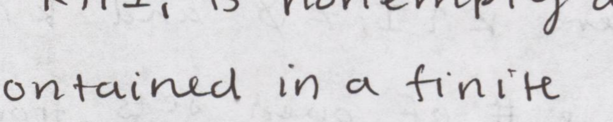
We prove this by contradiction.

Suppose that K is not contained in any finite subcover from G.

By hypothesis K is bounded.

So $K \subseteq [-r, r]$ for some $r > 0$

Let $I_1 = [-r, r]$



Bisect I_1 into two intervals $I_1' = [r, 0]$ and $I_1'' = [0, r]$

Then at least one of $K \cap I_1'$ or $K \cap I_1''$ is nonempty and has the property that it is not contained in a finite subcover from G.

(For if both $K \cap I_1'$ and $K \cap I_1''$ are contained in finite subcovers from G you could put those two finite subcovers together and then finitely cover K.)

Let

$$I_2 = \begin{cases} I_1' & \text{if } K \cap I_1' \neq \emptyset \text{ and can't be covered by a finite \# of elements from } G. \\ I_1'' & \text{if } K \cap I_1'' \neq \emptyset \text{ and can't be covered by a finite \# of elements from } G. \end{cases}$$

Now repeat this procedure on I_2.

Bisect I_2 into two closed intervals:

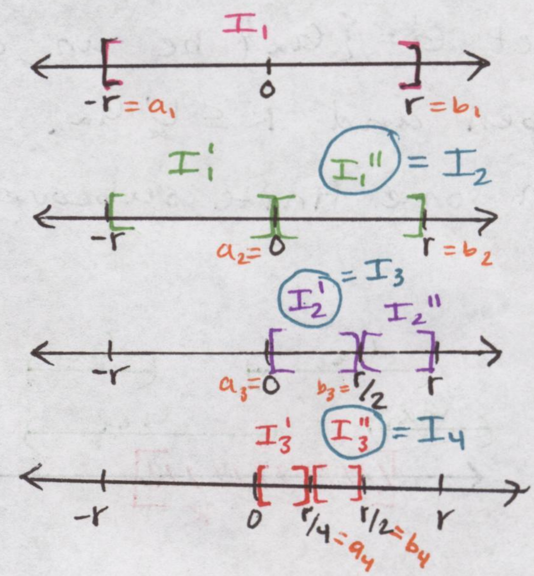
I_2' and I_2''

Again one of $K \cap I_2'$ or $K \cap I_2''$

is nonempty and can't be covered

by a finite # of elements from G.

(same reasoning as first step)



Let

$$I_3 = \begin{cases} I_2' & \text{if } K \cap I_2' \neq \emptyset \text{ and can't be covered by a finite \# of elements from } G. \\ I_2'' & \text{if } K \cap I_2'' \neq \emptyset \text{ and can't be covered by a finite \# of elements from } G. \end{cases}$$

Continue this process forever and ever and ever...

to get a nested sequence

$$\ldots \subseteq I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 = [-r, r]$$

where $K \cap I_i \neq \emptyset$ and $K \cap I_i$ can't be covered by a finite # of open sets from G.

Claim: There exists $\mathcal{S} \in \mathbb{R}$ with $\mathcal{S} \in \bigcap_{n=1}^{\infty} I_n$

proof of claim: Let $I_n = [a_n, b_n]$

since the intervals are nested within I_1, we have

$a_n \leq b_1 = r$ for all n.

So the sequence (a_n) is bounded from above

Thus, $\mathcal{S} = \sup\{a_n \mid n \geq 1\}$ exists. Thus, $a_n \leq \mathcal{S}$ for all n.

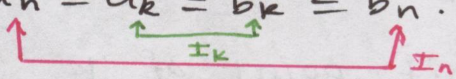
Let's show $\mathcal{S} \leq b_n$ for all n.

This is established by showing that for any particular n, b_n is an upper bound for $\{a_k \mid k \geq 1\}$, and hence $\mathcal{S} \leq b_n$ since $\mathcal{S}$ is the least upper bound of $\{a_k \mid k \geq 1\}$.

Let n be fixed.

case (i) if $n \leq k$, then since $I_k \subseteq I_n$

we have $a_n \leq a_k \leq b_k \leq b_n$. So $a_k \leq b_n$



case (ii)

if $k < n$, then since $I_n \subseteq I_k$ we have

$a_k \leq a_n \leq b_n \leq b_k$. so $a_k \leq b_n$.

Thus $\mathcal{S} \leq b_n$ for all n.

Therefore, $a_n \leq \mathcal{S} \leq b_n$ for all n. Thus $\mathcal{S} \in \bigcap_{n=1}^{\infty} I_n$ claim

Claim: $\mathcal{S} = \inf\{b_k \mid k \geq 1\}$

proof of claim: since $-r \leq b_k$ for all k, $x = \inf\{b_k \mid k \geq 1\}$ exists.

Let's show that $x = \mathcal{S}$.

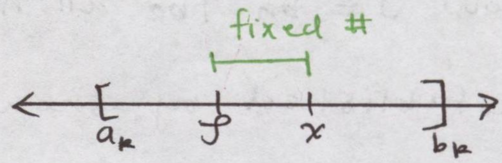
We know that $\mathcal{S} \leq b_k$ for all k from earlier.

So $\mathcal{S}$ is a lower bound on $\{b_k \mid k \geq 1\}$, thus $\mathcal{S} \leq x$.

We now show $\mathcal{S} \geq x$ and hence $\mathcal{S} = x$.

prove this by contradiction

Suppose $\mathcal{S} < x$



Then $a_k \leq \mathcal{S} < x \leq b_k \ \forall k$.

by construction the length of $I_k = [a_k, b_k]$ is $\frac{r}{2^{k-2}}$

and $\frac{r}{2^{k-2}} \to 0$ as $k \to \infty$.

so there is some k_0 where the length of $I_k = [a_k, b_k]$ is smaller than $|x - \mathcal{S}|$. That can't happen since x and $\mathcal{S}$ are both in $I_k \ \forall k$. contradiction! claim

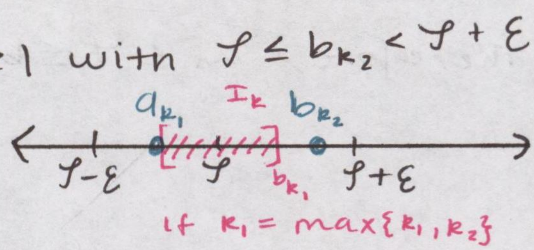
Claim $\mathcal{S}$ is a limit point of K

proof of claim: let $\varepsilon > 0$.

since $\mathcal{S} = \sup\{a_k \mid k \geq 1\}$ there exists $k_1 \geq 1$ where

$\mathcal{S} - \varepsilon < a_{k_1} \leq \mathcal{S}$ useful inf/sup fact.

since $\mathcal{S} = \inf\{b_k \mid k \geq 1\}$ there exists $k_2 \geq 1$ with $\mathcal{S} \leq b_{k_2} < \mathcal{S} + \varepsilon$



Let $k = \max\{k_1, k_2\}$.

Then, $\mathcal{S} - \varepsilon < a_{k_1} \leq a_k \leq \mathcal{S} \leq b_k \leq b_{k_2} < \mathcal{S} + \varepsilon$

So, $I_k = [a_k, b_k] \subseteq (\mathcal{S} - \varepsilon, \mathcal{S} + \varepsilon)$

We know $I_k \cap K \neq \emptyset$ and there exists some point

P.3 11/14

k in $I_k \cap K$ that isn't γ

(If $\gamma \in K$ and $I_k \cap K = \{\gamma\}$ then pick some G_α from G that has $\gamma \in G_\alpha$ and G_α would be a finite subcover of $I_k \cap K$. Which can't happen)

So there exists a point from K in $(\gamma - \varepsilon, \gamma + \varepsilon)$ that isn't γ.

claim