note: ways to prove that some bound on a set is the inf/sup of the set

(1) use useful inf/sup fact

(2) Use def. Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$

- $b = \sup(S)$ iff

  (i) $x \leq b$ for $x \in S$

  (ii) $b \leq c$ $\forall$ upper bounds $c$ of $S$

- $b = \inf(S)$ iff

  (i) $b \leq x$ for $x \in S$

  (ii) $c \leq b$ $\forall$ lower bounds $c$ of $S$

## Completeness Axiom

Let $S \subseteq \mathbb{R}$ with $S \neq \emptyset$

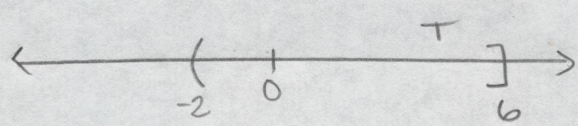
If $S$ is bounded from above, then the supremum of $S$ exists.

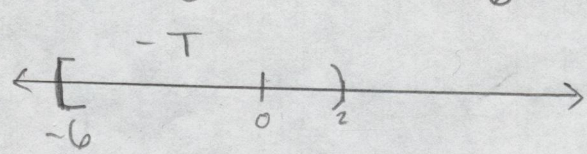
**Theorem:** If $T \subseteq \mathbb{R}$ and $T \neq \emptyset$

If $T$ is bounded from below then the infimum of $T$ exists

**Proof:** Let $T \subseteq \mathbb{R}$, $T \neq \emptyset$, and $T$ be bounded from below

~~Ex.~~ $T = (-2, 6]$




Let $-T = \{-x \mid x \in T\}$

suppose $b$ is a lower bound of $T$ (by assumption $b$ exists)

So, $b \leq x \quad \forall x \in T$

Then, $-x \leq -b \quad \forall x \in T$

So, $-b$ is an upper bound of $-T$

By the completeness axiom, the supremum of $-T$ exists.

Let $b_{-T} = \sup(-T)$.

So,

- $-x \leq b_{-T} \quad \forall x \in T$
- $b_{-T} \leq c \quad \forall$ upper bounds $c$ of $-T$

Set $b_T = -b_{-T}$

Claim: $b_T = \inf(T)$

- We know that $-x \leq b_{-T}$ for all $x \in T$

So, $b_T = -b_{-T} \leq x \quad \forall x \in T$

So, $b_T$ is a lower bound for $T$.

- Let $d$ be another lower bound for $T$

Then $d \leq x \quad \forall x \in T$

So, $-x \leq -d \quad \forall x \in T$

So, $-d$ is an upper bound of $-T$

Thus $b_{-T} \leq -d$

Hence $-b_{-T} \geq d$

So, $d \leq b_T$. Hence $b_T$ is the greatest lower bound of $T$.

$\boxtimes$

# Well-ordering Principle

Every non-empty subset of $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ contains a least element.

**Example:**

$$S = \{5, 7, 9, 11, \dots\} \subseteq \mathbb{N}$$

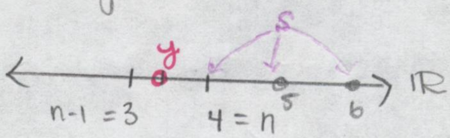
$\uparrow$ least element

**Lemma:**

Let $y \in \mathbb{R}$ with $y > 0$

Then $\exists\, n \in \mathbb{N}$ with $n-1 \le y < n$

**Example:** Let $y = \pi \approx 3.14\dots$




Then $n-1 = 3$ & $n = 4$

**Proof:** Let $S = \{m \in \mathbb{N} \mid y < m\}$

By the Archimedean principle $S \neq \emptyset$

By the well-ordering principle, $S$ has a least element. Call it $n$, then, $n-1 \notin S$

so $\underbrace{n-1 \le y}_{n-1 \notin S}$ $\underbrace{y < n}_{n \in S}$ $\boxtimes$

# Density Theorem:

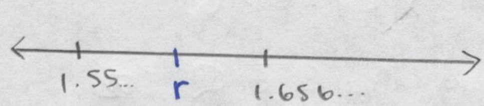
Let $x, y \in \mathbb{R}$ with $x < y$

Then $\exists\, r \in \mathbb{Q}$ with $x < r < y$

Note: $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

$\mathbb{Q}$ = fractions = rational #s

**Example:** $x = 1.55\dots$ $y = 1.656\dots$




$r = 1.57 = \frac{157}{100}$

proof:

$\boxed{\text{Part 1}}$ Suppose $0 < x < y$

so, $0 < y - x$.

so, $0 < \frac{1}{y-x}$

By the Archimedean Principle

$\frac{1}{y-x} < n$ for some $n \in \mathbb{N}$

Thus, $\frac{1}{n} < y - x$

so, $1 < ny - nx$

Then, $\underline{1 + nx < ny}$

Since $nx > 0$, we can apply the lemma to obtain $m \in \mathbb{N}$ with $m-1 \leq \underline{nx < m}$

Thus, $m \leq nx + 1 < ny$.

$m - 1 \leq nx$ From above

so, $nx < m < ny$

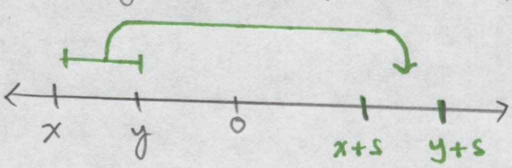
Thus $x < \frac{m}{n} < y$. Set $r = \frac{m}{n}$

$\boxed{\text{Part 2}}$ what if $x < 0$

Let $s \in \mathbb{N}$ with $0 < x + s$

Apply $\boxed{\text{Part 1}}$ to $0 < x + s < y + s$

to get $r \in \mathbb{Q}$ with $x + s < r < y + s$




so $x < r - s < y$

And $r - s \in \mathbb{Q}$ ◪

# Absolute Value

**Def:** Let $x \in \mathbb{R}$, the absolute value of $x$ is

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|5| = 5$

$|-10| = -(-10) = 10$

**Facts** let $a, b \in \mathbb{R}$, then:

(1) $|ab| = |a||b|$

(2) Let $c > 0$ then $|a| \leq c$ iff $-c \leq a \leq c$

(3) $-|a| \leq a \leq |a|$

(4) $|a+b| \leq |a| + |b|$ triangle inequality

(5) $\big||a| - |b|\big| \leq |a-b|$

(6) $|a-b| \leq |a| + |b|$

**proof:** (1) and (5) are homework problems

(2) ($\Rightarrow$) Suppose $|a| \leq c$

Case 1: If $a \geq 0$, then $|a| = a$

so, $0 \leq a \leq c$

then $-c \leq 0 \leq a \leq c$ so $-c \leq a \leq c$

Case 2: If $a < 0$, then $|a| = -a > 0$

then $-a \leq c$

so $-c \leq a$

since $a < 0$ and $c > 0$, we have $-c \leq a < 0 < c$

hence $-c \leq a \leq c$

So in either case $-c \leq a \leq c$

($\Leftarrow$) Suppose $-c \le a \le c$ where $c > 0$

Then $-c \le a$ and $a \le c$

so $-a \le c$ and $a \le c$

then, $|a| \le c$ $\square$

**proof (3)**

If $a=0$, then $-|a| \le a \le |a|$

If $a \neq 0$, set $c = |a| > 0$

Since $|a| \le c$, by (2) we have $-c \le a \le c$

so $-|a| \le a \le |a|$ $\square$

**proof (4)** Triangle ineq.

From part (3) we have $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$

By adding we get

$$-(|a|+|b|) \le a+b \le |a|+|b|$$

By part (2), with $c = |a|+|b|$ we get $|a+b| \le |a|+|b|$ $\square$

**proof (6)**

By part (4)

$$|a-b| = |a+(-b)| \le |a| + |-b|$$
$$= |a| + |b|$$

$\square$

**Note:** Similarly to part (2)

you can show $-c < a < c$ iff $|a| < c$

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**Corollary:** Let $x, y, \varepsilon \in \mathbb{R}$ with $\varepsilon > 0$

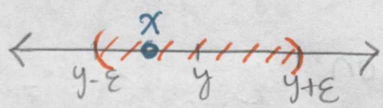
Then $|x-y| < \varepsilon$ iff $\varepsilon < x-y < \varepsilon$ iff $y-\varepsilon < x < y+\varepsilon$

similarly, $|x-y| \leq \varepsilon$ iff $\varepsilon \leq x-y \leq \varepsilon$ iff $y-\varepsilon \leq x \leq y+\varepsilon$

**proof** use part (2) and the previous "note" $\boxtimes$

with $a = x-y$ & $c = \varepsilon$

Picture of $|x-y| < \varepsilon$




Done with Hw 1 stuff

## Limits of Sequences

**Def:** A sequence of real numbers is an ordered list of real numbers. We write the $n^{th}$ term in the list using subscripts such as $a_n$.

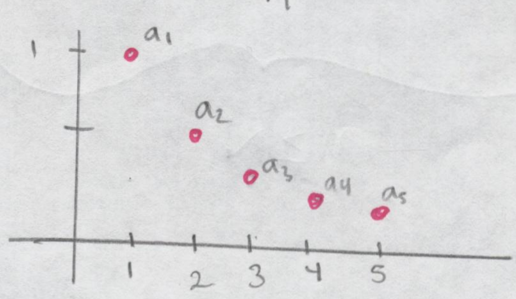
Sometimes we write $(a_n)$ or $(a_n)_{n=1}^{\infty}$ for a sequence.

Another way to define a sequence is:

as a function $f: \mathbb{N} \to \mathbb{R}$

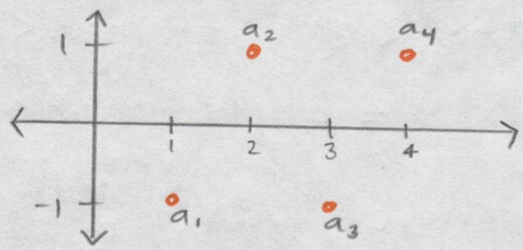
where $f(n) = a_n$

**Example:** $a_n = \frac{1}{n}$ sequence: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$



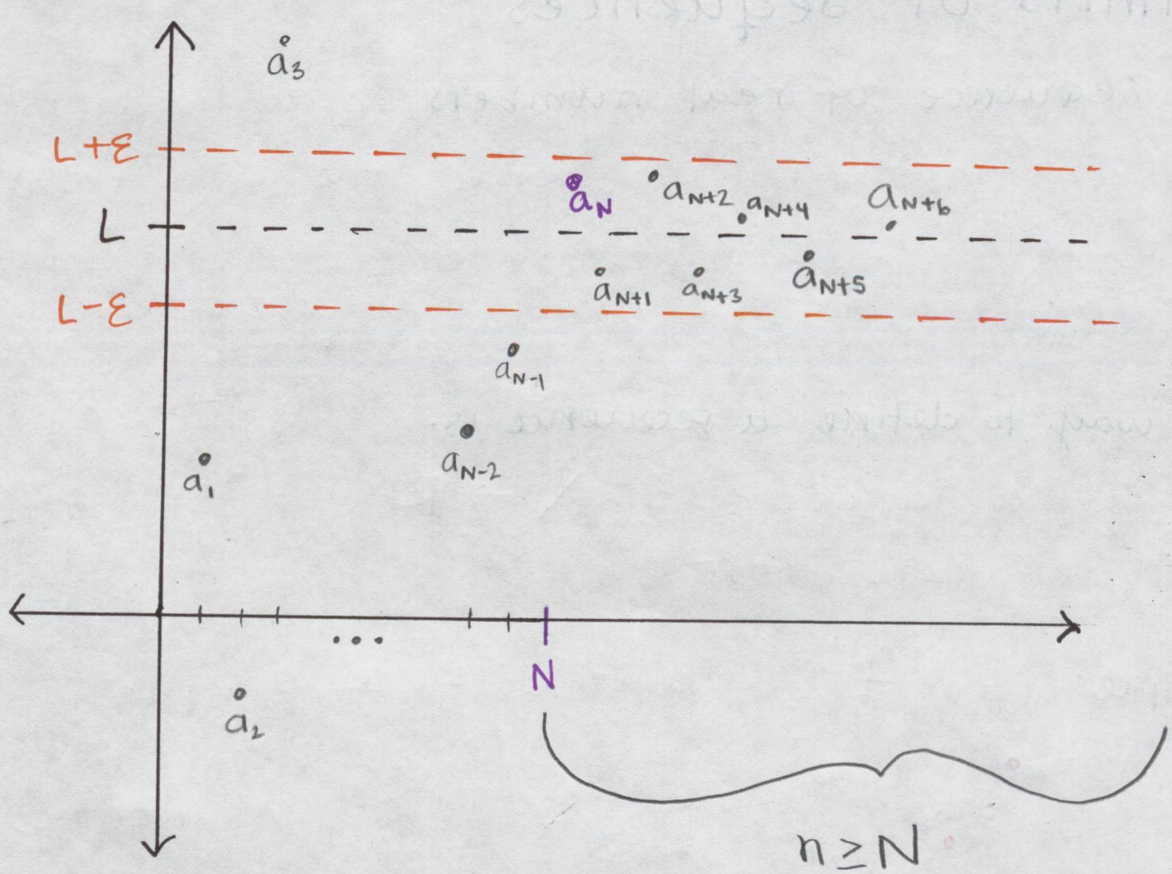
Example: $a_n = (-1)^n$

sequence: $-1, 1, -1, 1, \ldots$




Def: Let $(a_n)$ be a sequence of real numbers. We say that $(a_n)$ **converges** to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < \varepsilon$.

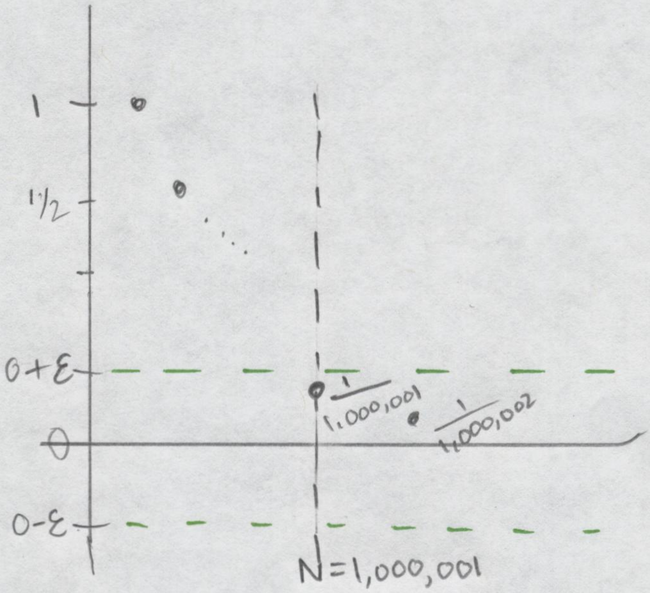
notation: $\lim_{n\to\infty} a_n = L$




N depends on $\varepsilon$

Example: $a_n = \frac{1}{n}$ I think $L = 0$

Let $\varepsilon = \frac{1}{1{,}000{,}000} = 0.000001$




Set $N = 1{,}000{,}001$

If $n \geq N$ then

$$\left|\frac{1}{n} - 0\right| < \frac{1}{1{,}000{,}000} = \varepsilon$$