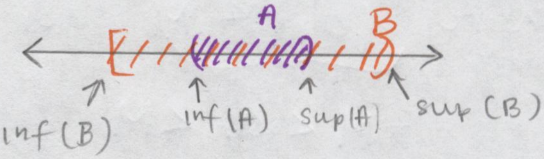
HW 1 #6 Let A and B be subsets of $\mathbb{R}$. Suppose A and B are bounded from above and below. Suppose

Suppose $A \subseteq B$



(a) Prove $\sup(A) \le \sup(B)$ and $\inf(B) \le \inf(A)$

proof: Let $S_A = \sup(A)$ and $S_B = \sup(B)$

By def of supremum,

- $a \le S_A \;\forall\, a \in A$
- $S_A \le c$ for any other upper-bounds c of A

Since S_B is the supremum of B

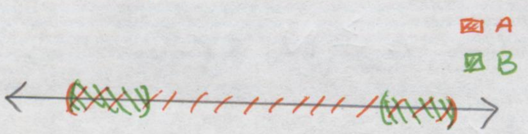
$$b \le S_B \;\forall\, b \in B$$

Since $A \subseteq B$, $a \le S_B \;\forall\, a \in A$. So, S_B is an upper bound of A. So $S_A \le S_B$ (Do inf part as practice)

(b) If $\sup(A) = \sup(B)$ and $\inf(A) = \inf(B)$ do we necessarily have $A = B$?

No! Let $A = (0,10)$

$B = (0,2) \cup (8,10)$



counter-example $\sup(A) = \sup(B) = 10$

$\inf(A) = \inf(B) = 0$ but $A \ne B$

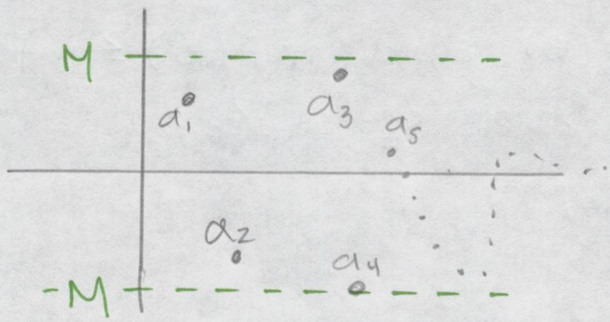
another example: $A = (-2, 5)$ but $A \ne B$

$B = [-2, 5]$

Limits continued...

Def: A sequence (a_n) is **bounded** if there exists $M > 0$ where $\underbrace{|a_n| < M}$ for all n.

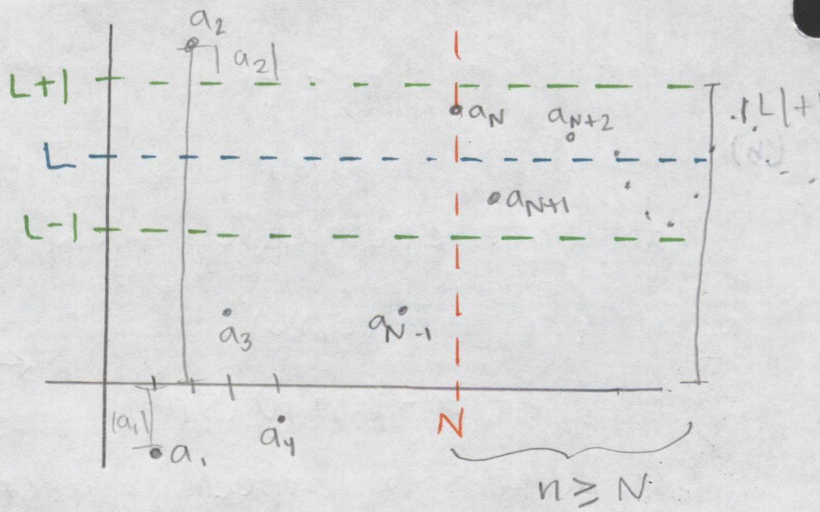
$$-M < a_n < M$$



Another way: (a_n) is bounded if $\exists\ M > 0$ and $N < 0$ where $N < a_n < M \ \forall n$.

Theorem: If (a_n) converges, then (a_n) is bounded.

proof: Let $L = \lim_{n \to \infty} a_n$



Let $\varepsilon = 1$

By def of limit $\exists\ N > 0$ where if $n \geq N$ then

$|a_n - L| < 1$

so if $n \geq N$ then,

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L|$$

Let $M = \max\{\underbrace{|a_1|, |a_2|, \ldots, |a_{N-1}|}_{\text{max of the bound } a_n \text{ where } 1 \leq n \leq N-1}, \underbrace{|L|+1}_{\text{Bounds } a_n \text{ where } n \geq N}\}$

Then $|a_n| \leq M$ for all n. $\boxtimes$

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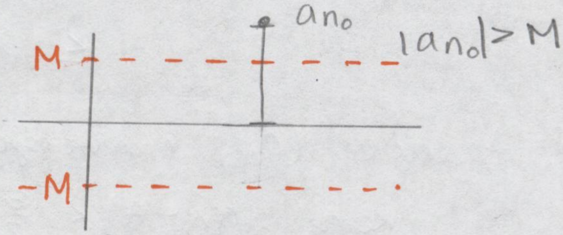
Contrapositive: If (a_n) is unbounded then (a_n) diverges

- **Def:** (a_n) is **unbounded** if for every $M > 0$

  $\exists\, n_0$ where $|a_{n_0}| > M$



Example: $a_n = n^5$ diverges

we show this by showing (n^5) is unbounded

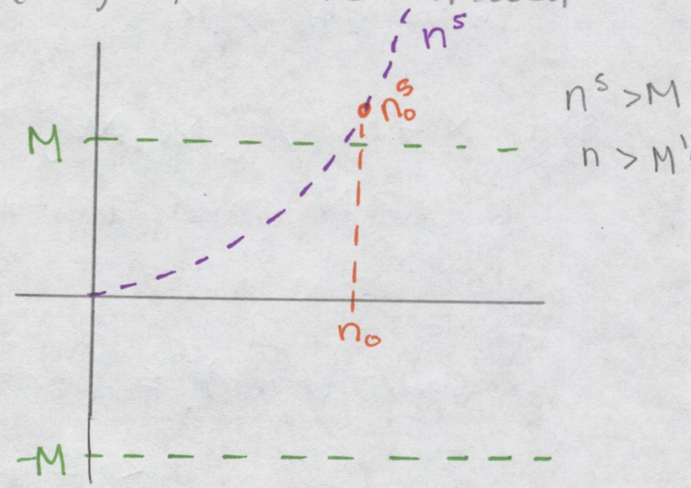
let $M > 0$

Pick $n_0 > M^{1/5}$

Then $n_0^5 > M$

So $|n_0^5| > M$

Thus, (n^5) is unbounded ☒



Theorem: Let (a_n) and (b_n) be convergent sequences with limits A and B respectively.

Then,
(1) $(a_n + b_n)$ converges to $A + B$
(2) $(a_n b_n)$ converges to AB
(3) If $A \neq 0$ and $a_n \neq 0\ \forall n$, then $\left(\frac{1}{a_n}\right)$ converges to $\frac{1}{A}$
(4) If $\alpha \in \mathbb{R}$, then (αa_n) converges to αA

note that:

(1) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} (a_n) + \lim\limits_{n\to\infty} (b_n)$

(2) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} (a_n) \cdot \lim\limits_{n\to} (b_n)$

(3) $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim\limits_{n\to\infty} a_n}$

(4) $\lim\limits_{n\to\infty} (\alpha a_n) = \alpha \left(\lim\limits_{n\to\infty} a_n\right)$

proof of (1): Let $\varepsilon > 0$

$\lim_{n\to\infty} c_n = C$ if for every $\varepsilon > 0$ $\exists N$ such that $n \geq N \Rightarrow |c_n - C| < \varepsilon$

Note that,

$$|(a_n+b_n)-(A+B)| = |(a_n-A)+(b_n-B)|$$
$$\leq |a_n-A| + |b_n-B|$$

- since (a_n) converges to A $\exists N_1 > 0$ where if $n \geq N_1$ then $|a_n - A| < \frac{\varepsilon}{2}$
- since (b_n) converges to B $\exists N_2 > 0$ where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2}$

Set $N = \max\{N_1, N_2\}$

If $n \geq N$ then $|a_n - A| < \frac{\varepsilon}{2}$ and $|b_n - B| < \frac{\varepsilon}{2}$

so if $n \geq N$ then

$$|(a_n+b_n)-(A+B)| \leq |a_n-A| + |b_n-B|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

proof of (2): Let $\varepsilon > 0$. Since (b_n) converges, it is bounded so $\exists \underline{M > 0}$ where $|b_n| \leq M$ $\forall n$.

Note that,
$$|a_n b_n - AB| = |a_n b_n - b_n A + b_n A - AB|$$
$$\leq |a_n b_n - b_n A| + |b_n A - AB|$$
$$= |b_n|\,|a_n - A| + |A|\,|b_n - B|$$
$$\leq M \cdot |a_n - A| + |A|\,|b_n - B|$$
$$< \underbrace{M}_{\text{not } 0} |a_n - A| + \underbrace{(|A|+1)}_{\text{not } 0} |b_n - B|$$

Since (a_n) converges to A, $\exists N_1 > 0$ where $n \geq N_1$ then $|a_n - A| < \frac{\varepsilon}{2M}$. Since (b_n) converges to B, $\exists N_2 > 0$ where $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2(|A|+1)}$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then,

$$|a_n b_n - AB| < M|a_n - A| + (|A|+1)|b_n - B|$$
$$< M \frac{\varepsilon}{2M} + (|A|+1) \cdot \frac{\varepsilon}{2(|A|+1)} = \varepsilon \quad \square$$

HW 2 #3(e) Show that $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$

Recall:
$\lim_{n\to\infty} a_n = L$
if $\forall\, \varepsilon > 0\ \exists\, N > 0$
where if $n \geq N$
then $|a_n - L| < \varepsilon$

Proof: Let $\varepsilon > 0$

Note that:

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| = \left| \frac{2n^2 - 2n^2 - 1}{2(n^2+1)} \right| = \left| \frac{-1}{4n^2+2} \right| = \frac{1}{4n^2+2}$$

and $\frac{1}{4n^2+2} < \varepsilon$ iff $\frac{1}{\varepsilon} < 4n^2+2$ iff $\frac{1}{4\varepsilon} - \frac{1}{2} < n^2$ iff $\sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}} < n$

Since $n > 0$

Let $N > \sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}}$ then if $n > N$, then $\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon$

Another way: Let $\varepsilon > 0$

note that: $\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| = \frac{1}{4n^2+2} < \frac{1}{4n^2} < \frac{1}{n^2} \leq \frac{1}{n}$

And $\frac{1}{n} < \varepsilon$ iff $\frac{1}{\varepsilon} < n$.

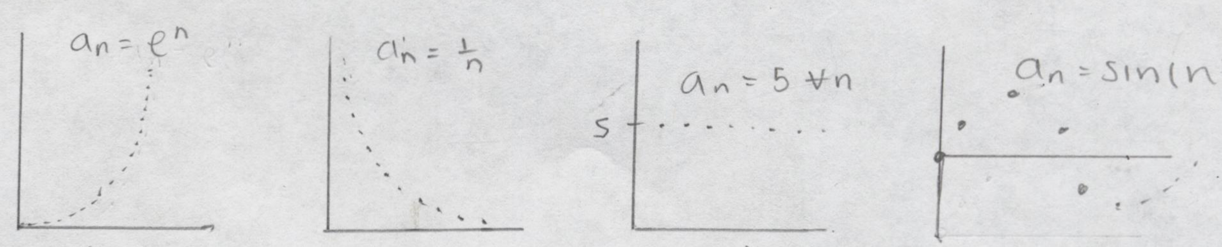
Set $N > \frac{1}{\varepsilon}$

If $n \geq N$, then $\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon$. $\boxtimes$

Monotone Convergence Thm (still on HW 2)

Def:
- Let (a_n) be a sequence of real numbers. We say that (a_n) is **monotonically increasing** if $a_n \leq a_{n+1}\ \forall n$.
- We say that (a_n) is **monotonically decreasing** if $a_{n+1} \leq a_n\ \forall n$.
- We say that (a_n) is **monotone** if (a_n) is monotonically increasing or monotonically decreasing.

Ex:



- mon. inc.
- monotone

- mon. dec.
- monotone

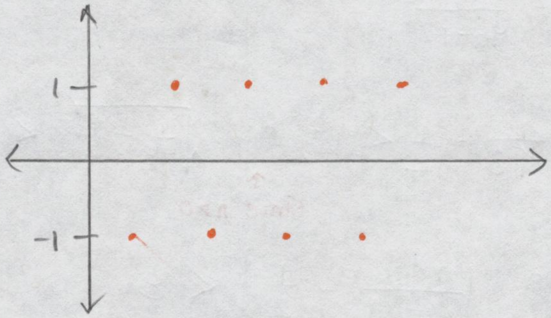
- mon. inc
- mon. dec

} Both

Not Monotone

In general, convergence and boundedness don't imply each other.

- If (a_n) converges then (a_n) is bounded
- Converse is not true: If (a_n) is bounded, it might not converge,



Ex: $(a_n) = (-1)^n$

is bounded, but does not converge.

Monotone Convergence Thm.

A monotone sequence of real numbers is convergent iff the sequence is bounded.

Furthermore:

(1) If (a_n) is bounded monotonically increasing sequence, then $\lim_{n\to\infty} a_n = \sup\{a_n \mid n=1,2,3,\dots\}$
$$= \sup\{a_1, a_2, a_3, \dots\}$$

(2) If (a_n) is a bounded monotonically decreasing sequence then $\lim_{n\to\infty} a_n = \inf\{a_n \mid n=1,2,3,\dots\}$
$$= \inf\{a_1, a_2, a_3, \dots\}$$

Ex: $a_n = \frac{1}{n}$ ← Bounded mon. dec.

$$\lim_{n\to\infty} \frac{1}{n} = \inf\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\right\} = 0$$

Proof: We prove this fact for a monotonically increasing sequence (a_n).

- If (a_n) converges, then (a_n) is bounded [Previous thm]

Suppose (a_n) is bounded

Since (a_n) is bounded, $\exists M > 0$ where

$$-M \leq a_n \leq M \quad \text{for all } n.$$

Let $S = \{a_n \mid n = 1, 2, 3, \dots\} = \{a_1, a_2, a_3, \dots\}$

Then M is an upper bound for S.

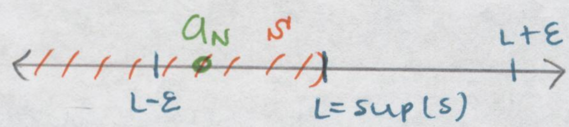
By the completeness axiom, the supremum of S exist

Let $L = \sup(S)$

Let $\varepsilon > 0$

By the useful sup/inf fact $\exists a_N \in S$ with

$$L - \varepsilon < a_N \leq L$$



Since (a_n) is monotonically increasing, if $n \geq N$ then $a_N \leq a_n$. So, if $n \geq N$, then $L - \varepsilon < a_N \leq a_n \underset{\substack{\uparrow \\ L = \sup(S)}}{\leq} L < L + \varepsilon$

So, if $n \geq N$, then $L - \varepsilon < a_n < L + \varepsilon$

Therefore, if $n \geq N$, then $|a_n - L| < \varepsilon$

So $\lim_{n \to \infty} a_n = L = \sup(S)$ ☒

Application (will not be on any exam!)

$f(x)=0$ iff $x=\pm\sqrt{a}$

let $a>0$, let's approximate $\sqrt{a}$

Newton's Method: Find a zero for $f(x)=0$. Pick a starting pt. x_1

Define: $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, $x_1=$ picked

Hopefully (x_n) converges to a root of $f(x)=0$

$f(x)=x^2-a$

$$x_{n+1}=x_n-\frac{x_n^2-a}{2x_n}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$$

$f'(x)=2x$

let $x_1>0$ be arbitrary

Define $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$ for $n\geq 1$

Let's show $\lim_{n\to\infty}x_n=\sqrt{a}$

$* \, 2x_n$

$2x_nx_{n+1}=x_n^2+a$

Note: $x_n>0$ for all n

claim 1: $x_n^2\geq a$ for all $n\geq 2$

Note that $x_n^2-2x_{n+1}x_n+a=0$ for $n\geq 1$

Hence, $x^2-2x_{n+1}+a=0$ has a real root for all $n\geq 1$

namely $x=x_n$.

The discrimant of $\dfrac{2x_{n+1}\pm\sqrt{4x_{n+1}^2-4a}}{2}$ must be non-negative

That is, $4x_{n+1}^2-4a\geq 0$.

So, $x_{n+1}\geq\sqrt{a}$ for all $n\geq 1$

claim 2: (x_n) is monotonically decreasing eventually.

when $n\geq 2$ we have that $x_n-x_{n-1}=x_n-\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)=\dfrac{x_n^2-a}{2x_n}\geq 0$

By claim 1 & notes

So, $x_n-x_{n+1}\geq 0$ for $n\geq 2$. So, $x_n\geq x_{n+1}$ for $n\geq 2$.

So (x_n) is monotonically decreasing after $n\geq 2$.

By the monotone convergence thm. (x_n) has a limit.

Let $A=\lim_{n\to\infty}x_n$. Since $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$ we have that

$$A=\lim_{n\to\infty}x_{n+1}=\frac{1}{2}\left(\lim_{x\to\infty}x_n+\frac{a}{\lim_{n\to\infty}x_n}\right)=\frac{1}{2}\left(A+\frac{a}{A}\right)$$

so, $A = \frac{1}{2}\left(A + \frac{a}{A}\right)$

Thus, $2A^2 = A^2 + a$

so, $A^2 = a$

thus, $A = \sqrt{a}$ [$A > 0$, since $x_n > 0 \ \forall n$]

Error Bound from calculation

$x_n \geq \sqrt{a} \quad \forall n \geq 2$

so, $x_n \geq \sqrt{a} \geq \frac{a}{x_n}$

$\uparrow$

$x_n \geq \sqrt{a} = \frac{a}{\sqrt{a}}$

$\sqrt{a} \geq \frac{a}{x_n}$

so, $|x_n - \sqrt{a}| \leq \frac{x_n^2 - a}{x_n}$

$\forall n \geq 2$

thus, $0 \leq x_n - \sqrt{a} \leq x_n - \frac{a}{x_n} = \frac{x_n^2 - a}{x_n}$

Table: $a = 2$

Estimate $\sqrt{a} = \sqrt{2}$ Pick $x_1 = 1$

x_{n+1}	Error Bound $(x_n^2 - 2)/x_n$
$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$	$\frac{x_2^2 - 2}{x_n} = \frac{(1.5)^2 - 2}{1.5} = \frac{1}{6} \approx 0.11666\ldots$
$x_3 = \frac{17}{12} \approx 1.41666\ldots$	$\frac{1}{204} \approx 0.004901\ldots$
$x_4 = \frac{1}{2}\left(x_3 + \frac{2}{x_3}\right) \approx 1.414215686.$	$\frac{1}{239416} \approx 0.000004247\ldots$