P.1 10/1 Monday Week 7

HW 2 #3(f) Show that $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n!} = 0$

proof: Let $\varepsilon > 0$

Note that $\left|\frac{\sqrt{n^2+1}}{n!} - 0\right| = \left|\frac{\sqrt{n^2+1}}{n!}\right| = \frac{\sqrt{n^2+1}}{n!} \underset{\substack{\uparrow\\ 1\le n\\ \text{so } 1\le n^2}}{\le} \frac{\sqrt{n^2+n^2}}{n!}$

$= \frac{\sqrt{2n^2}}{n!} = \frac{\sqrt{2}\,n}{n!} = \frac{\sqrt{2}\,\cancel{n}}{\cancel{n}(n-1)!} = \frac{\sqrt{2}}{(n-1)!} \underset{\uparrow}{\le} \frac{\sqrt{2}}{2^{n-2}}$

$(n-1)! = (n-1)(n-2)\cdots(3)(2)(1)$

If $n \ge 3$ then

$(n-1)! \ge \underbrace{2\cdot 2\cdot 2\cdots 2}_{n-2 \text{ of them}} = 2^{n-2}$

so if $n \ge 3$ then $\frac{1}{(n-1)!} \le \frac{1}{2^{n-2}}$

$n = 4$

$(n-1)! = 3\cdot 2\cdot 1$ (2)

$n = 5$

$(n-1)! = 4\cdot 3\cdot 2\cdot 1$ (3)

Suppose $n \ge 3$, Then $\frac{\sqrt{2}}{2^{n-2}} < \varepsilon$ iff $\frac{\sqrt{2}}{\varepsilon} < 2^{n-2}$

$\underset{\uparrow}{\text{iff}}$ $\log_2\left(\frac{\sqrt{2}}{\varepsilon}\right) < n-2$ iff $\log_2\left(\frac{\sqrt{2}}{\varepsilon}\right) + 2 < n$

$x < y$ iff $\log(x) < \log(y)$

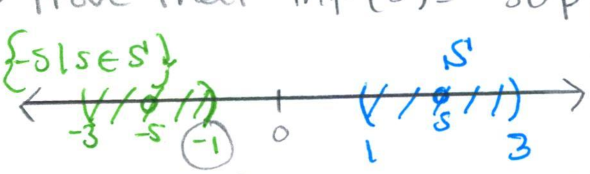
Set $N > \max\{3, \log_2\left(\frac{\sqrt{2}}{\varepsilon}\right) + 2\}$. If $n \ge N$, then

$\left|\frac{\sqrt{n^2+1}}{n!} - 0\right| \underset{\substack{\uparrow\\ n\ge 3}}{<} \frac{\sqrt{2}}{2^{n-2}} \underset{\substack{\uparrow\\ n \ge \log_2\left(\frac{\sqrt{2}}{\varepsilon}\right)+2}}{<} \varepsilon$ ▨

Test 1 (B) Let S be a non-empty subset of $\mathbb{R}$ that is bounded from below. Prove that $\inf(S) = -\sup\{-s \mid s \in S\}$

Ex: $S = (1,3)$

$\inf(S) = 1$



$-\sup\{-s \mid s \in S\} = -(-1) = 1 = \inf(S)$

Proof: Since $S \ne \emptyset$ and S is bounded from below, $\inf(S)$ exists, let $x = \inf(S)$

Let's show $-x = \sup\{-s \mid s \in S\}$

(1) Since $x = \inf(S)$ we know $x \le s$ $\forall s \in S$

Then $-x \ge -s$ $\forall s \in S$.

so $-x$ is an upper bound for $\{-s \mid s \in S\}$.

② Let's show that $-x$ is the least upper bound for $-S = \{-s \mid s \in S\}$

Let c be an upper bound for $-S$

Then, $-s \leq c \quad \forall\, s \in S$

so $\quad s \geq -c \quad \forall\, s \in S$

So $-c$ is a lower bound for S.

since $x = \inf(S)$, ie the greatest lower bound of S,

then $-c \leq x$.

so $c \geq -x$. so, $-x$ is the least upper bound of $-S$ ☒

② (another way to show that $-x = \sup(-S)$)

We already know from part ① that $-x$ is an upper bound for $-S$.

Let $\varepsilon > 0$

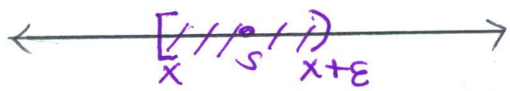
If we can find $s \in S$ with

$$-x - \varepsilon < -s \leq -x;$$

Then by the useful sup/inf fact $-x = \sup(-S)$

since $x = \inf(S)$ $\exists\, s \in S$ with $x \leq s < x + \varepsilon$

By the useful sup/inf fact



multiply by (-1) to get $-x - \varepsilon < -s \leq -x$ ☒

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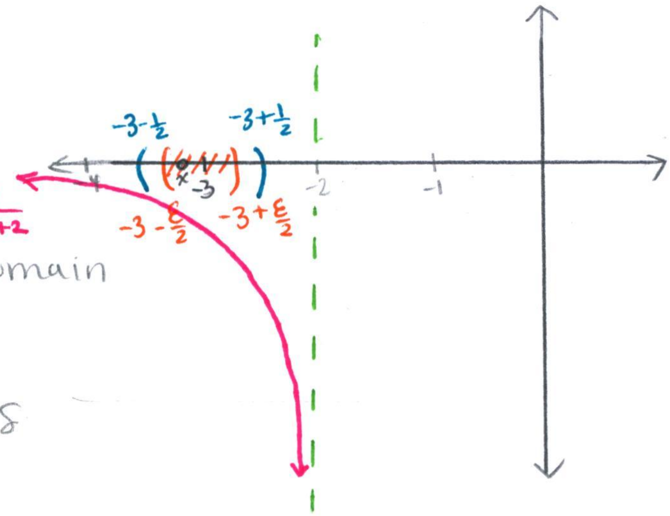
Limits Continued...

Ex: Prove that $\lim_{x \to -3} \frac{1}{x+2} = -1$.



Proof: let $\varepsilon > 0$

let $D = \mathbb{R} \setminus \{-2\}$ ← where D is the domain

We need to find $\delta > 0$

where if $x \in D$ and $0 < |x-(-3)| < \delta$

then $\left| \frac{1}{x+2} - (-1) \right| < \varepsilon$

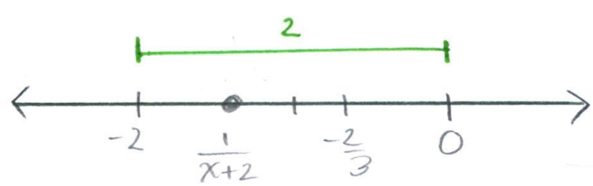
Note that, $\left| \frac{1}{x+2} \underbrace{-(-1)}_{\frac{x+2}{x+2}} \right| = \left| \frac{1+(x+2)}{x+2} \right| = \left| \frac{x+3}{x+2} \right|$

$= \underbrace{|x+3|}_{\text{we can control/bound with } \delta} \cdot \underbrace{\frac{1}{|x+2|}}_{\text{need to get rid of this guy by using a standing bound on } \delta.}$

suppose $\delta \le \frac{1}{2}$

let's try to bound $\frac{1}{|x+2|}$

If $\underbrace{|x+3|}_{|x-(-3)|} < \frac{1}{2}$, then $-\frac{1}{2} < x+3 < \frac{1}{2}$ so, $-\frac{3}{2} < x+2 < \frac{-1}{2}$



Then, $-\frac{2}{3} > \frac{1}{x+2} > -2$

Summarizing, if $|x+3| < \frac{1}{2}$, then $\left|\frac{1}{x+2}\right| = \frac{1}{x+2} < 2$

If $|x+3| < \frac{1}{2}$, then

$$\left| \frac{1}{x+2} - (-1) \right| = |x+3| \cdot \frac{1}{|x+2|} < 2|x+3|$$

Let $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. If $0 < |x+3| < \delta$, then

$$\left| \frac{1}{x+2} - (-1) \right| \underset{\uparrow \, |x+3| < \frac{1}{2}}{<} 2|x+3| \underset{\uparrow \, |x+3| < \frac{\varepsilon}{2}}{<} 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

$\boxtimes$

Practice:
#4

If $\boxed{\begin{array}{l}\lim_{n\to\infty} a_n = 0 \\ \lim_{n\to\infty} b_n = B \\ \alpha \in \mathbb{R}, \alpha \neq 0\end{array}} \implies \lim_{n\to\infty}(-3a_n + \alpha b_n + 5) = \alpha B + 5$

Proof: let $\varepsilon > 0$

Note that: $|(-3a_n + \alpha b_n + 5) - (\alpha B + 5)| = |-3a_n + \alpha b_n - \alpha B|$

$$= |-3(a_n - 0) + \alpha(b_n - B)|$$

$$\leq |-3(a_n-0)| + |\alpha(b_n - B)| \underset{|xy|=|x||y|}{=} 3\underbrace{|a_n - 0|}_{<\frac{\varepsilon}{3\cdot 2}} + \alpha \underbrace{|b_n - B|}_{<\frac{\varepsilon}{|\alpha|\cdot 2}}$$

goal

- Since $\lim_{n\to\infty} a_n = 0$, $\exists N_1 > 0$ where if $n \geq N_1$ then $|a_n - 0| < \frac{\varepsilon}{3\cdot 2}$
- Since $\lim_{n\to\infty} b_n = B$, $\exists N_2 > 0$ where if $n > N_2$ then $|b_n - B| < \frac{\varepsilon}{|\alpha|\cdot 2}$

Let $N = \max\{N_1, N_2\}$

If $n > N$, then $|(-3a_n + \alpha b_n + 5) - (\alpha B + 5)| \leq 3|a_n - 0| + |\alpha||b_n - B|$

$$< 3\cdot\frac{\varepsilon}{3\cdot 2} + |\alpha|\frac{\varepsilon}{2\cdot|\alpha|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \boxtimes$$

* if α could be 0

then $< 3|a_n - 0| + (|\alpha|+1)|b_n - B|$

$$< 3\cdot\frac{\varepsilon}{2\cdot 3} + (|\alpha|+1)\cdot\frac{\varepsilon}{2(|\alpha|+1)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$, and $h: D \to \mathbb{R}$

Let a be a limit point of D. Let $\alpha \in \mathbb{R}$.

Suppose that $\lim_{x\to a} f(x) = F$, $\lim_{x\to a} g(x) = G$, and $\lim_{x\to a} h(x) = H$

further suppose that $h(x) \neq 0 \ \forall x \in D$ and $H \neq 0$.

Then, (1) $\lim_{x\to a} \alpha = \alpha$

(2) $\lim_{x\to a} [f(x) + g(x)] = F + G$

(3) $\lim_{x\to a} [f(x) - g(x)] = F - G$

(4) $\lim_{x\to a} (\alpha f(x)) = \alpha F$

(5) $\lim_{x\to a} (f(x) g(x)) = FG$ ← HW

(6) $\lim_{x\to a} \frac{f(x)}{h(x)} = \frac{F}{H}$

proof part (2):

Let $\varepsilon > 0$

Note that: $|(f(x)+g(x))-(F+G)| = |(f(x)-F)+(g(x)-G)|$

$\leq |f(x)-F| + |g(x)-G|$

since $\lim_{x\to a} f(x) = F$, $\exists\ \delta_1 > 0$ where if $0<|x-a|<\delta_1$ then $|f(x)-F| < \frac{\varepsilon}{2}$

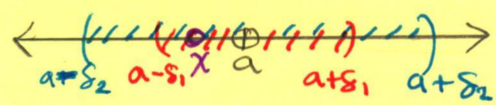
since $\lim_{x\to a} g(x) = G$, $\exists\ \delta_2 > 0$ where if $0<|x-a|<\delta_2$ then $|g(x)-G| < \frac{\varepsilon}{2}$

Let $\delta = \min\{\delta_1, \delta_2\}$

Then if $0<|x-a|<\delta$, then

$|(f(x)+g(x))-(F+G)| \leq |f(x)-F| + |g(x)-G|$

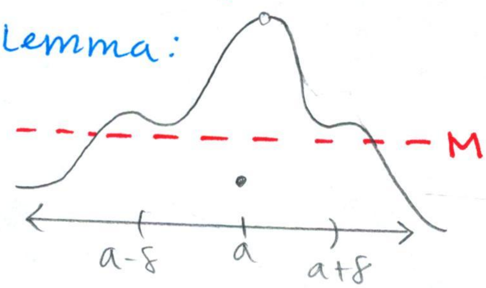
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ☒



$\delta_1 = \min\{\delta_1, \delta_2\}$
↑
Just for this pic.

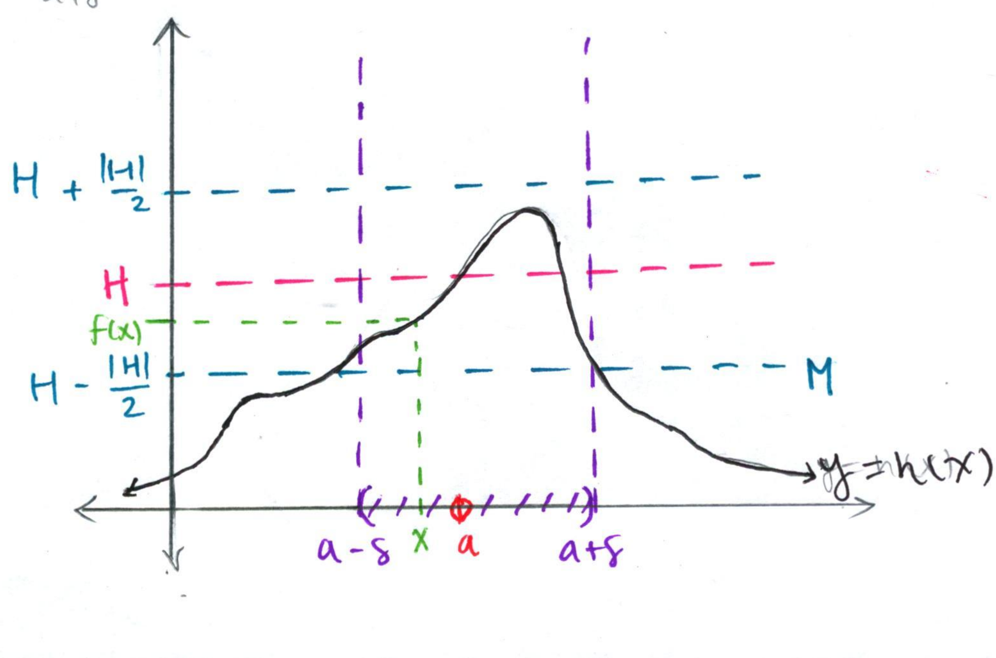
proof part (6):

Lemma:



Suppose $\lim_{x\to a} h(x) = H \neq 0$

Then $\exists\ \delta > 0$ and $M > 0$ where if $0<|x-a|<\delta$ and $x \in D$, then $|h(x)| > M$



Proof part (6):

Let $\varepsilon = \frac{|H|}{2} > 0$ (since $H \neq 0$)

Since $\lim_{x \to a} h(x) = H$, $\exists \delta > 0$ where if $0 < |x-a| < \delta$ and $x \in D$, then

$|h(x) - H| < \frac{|H|}{2}$

So if $0 < |x-a| < \delta$ and $x \in D$, then

$$|H| = |H - h(x) + h(x)| \leq |H - h(x)| + |h(x)| < \frac{|H|}{2} + h(x)$$

so, if $0 < |x-a| < \delta_{\vee x \in D}$ then $|H| < \frac{|H|}{2} + |h(x)|$

Thus, if $0 < |x-a| < \delta_{, \vee x \in D}$ then $\frac{|H|}{2} < |h(x)|$

set $M = \frac{|H|}{2}$. ☒

Now we prove (6) using the lemma.

let $\varepsilon > 0$

Note that

$$\left| \frac{1}{h(x)} - \frac{1}{H} \right| = \left| \frac{H - h(x)}{h(x) \cdot H} \right| = \frac{|H - h(x)|}{|h(x) \cdot H|} = \frac{|h(x) - H|}{|h(x)||H|}$$

By the lemma $\exists \delta_1 > 0$ and $M > 0$ where if $0 < |x-a| < \delta_1$ and $x \in D$ then $|h(x)| > M$

Since $\lim_{x \to a} h(x) = H$ $\exists \delta_2 > 0$ where if $0 < |x-a| < \delta_2$ and $x \in D$

then $|h(x) - H| < \varepsilon \cdot M \cdot |H|$

let $\delta = \min\{\delta_1, \delta_2\}$ Then if $0 < |x-a| < \delta$ and $x \in D$, then

$$\left| \frac{1}{h(x)} - \frac{1}{H} \right| = \frac{|h(x) - H|}{|h(x)||H|} \underset{\uparrow}{<} \frac{|h(x) - H|}{M \cdot |H|} < \frac{\varepsilon \cdot M \cdot |H|}{M \cdot |H|} = \varepsilon$$ ☒

$|h(x)| > M$

so $\frac{1}{|h(x)|} < \frac{1}{M}$