RISES, LEVELS, DROPS AND "+" SIGNS IN COMPOSITIONS: EXTENSIONS OF A PAPER BY ALLADI AND HOGGATT

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1. INTRODUCTION

A composition of n consists of an ordered sequence of positive integers whose sum is n. A palindromic composition (or palindrome) is one for which the sequence reads the same forwards and backwards. We derive results for the number of "+" signs, summands, levels (a summand followed by itself), rises (a summand followed by a larger one), and drops (a summand followed by a smaller one) for both compositions and palindromes of n. This generalizes a paper by Alladi and Hoggatt [1], where summands were restricted to be only 1s and 2s.

Some results by Alladi and Hoggatt can be generalized to compositions with summands of all possible sizes, but the connections with the Fibonacci sequence are specific to compositions with 1s and 2s. However, we will establish a connection to the Jacobsthal sequence [8], which arises in many contexts: tilings of a 3 x n board [7], meets between subsets of a lattice [3], and alternating sign matrices [4], to name just a few. Alladi and Hoggatt also derived results about the number of times a particular summand occurs in all compositions and palindromes of n, respectively. Generalizations of these results are given in [2].

In Section 2 we introduce the notation that will be used, methods to generate compositions and palindromes, as well as some easy results on the total numbers of compositions and palindromes, the numbers of "+" signs and the numbers of summands for both compositions and palindromes. We also derive the number of palindromes into i parts, which form an "enlarged" Pascal's triangle.

Section 3 contains the harder and more interesting results on the numbers of levels, rises and drops for compositions, as well as interesting connections between these quantities. In Section 4 we derive the corresponding results for palindromes. Unlike the case of compositions, we now have to distinguish between odd and even n. The final section contains generating functions for all quantities of interest.

2. NOTATION AND GENERAL RESULTS

We start with some notation and general results. Let

C_n, P_n	$=$	the number of compositions and palindromes of n, respectively
C_n^+, P_n^+	$=$	the number of "+" signs in all compositions and palindromes of n, respectively
C_n^{S}, P_n^{S}	$=$	the number of summands in all compositions and palindromes of n, respectively
$C_n(x)$	$=$	the number of compositions of n ending in x
$C_n(x,y)$	$=$	the number of compositions of n ending in $x+y$
r_n, l_n, d_n	$=$	the number of rises, levels, and drops in all compositions of n, respectively
$\tilde{r}_n$, $\tilde{l}_n$, $\tilde{d}_n$	$=$	the number of rises, levels, and drops in all palindromes of n, respectively.

We now look at ways of creating compositions and palindromes of n. Compositions of $n+1$ can be created from those of n by either appending '+1' to the right end of the composition or by increasing the rightmost summand by 1. This process is reversible and creates no duplicates, hence creates all compositions of $n+1$. To create all palindromes of n, combine a middle summand of size m (with the same parity as n, $0 \leq m \leq n$) with a composition of $\frac{n-m}{2}$ on the left and its mirror image on the right. Again, the process is reversible and creates no duplicates (see Lemma 2 of [2]). We will refer to these two methods as the *Composition Creation Method* (*CCM*) and the *Palindrome Creation Method* (*PCM*), respectively. Figure 1 illustrates the PCM.

6	7
1 4 1	1 5 1
2 2 2	2 3 2
1 1 2 1 1	1 1 3 1 1
3 3	3 1 3
1 2 2 1	1 2 1 2 1
2 1 1 2	2 1 1 1 2
1 1 1 1 1 1	1 1 1 1 1 1 1

Figure 1: Creating palindromes of $n = 6$ and $n = 7$

We can now state some basic results for the number of compositions, palindromes, "+" signs and summands.

Theorem 1 *1.* $C_n = 2^{n-1}$ *for* $n \geq 1$, $C_0 := 1$.

2. $P_{2k} = P_{2k+1} = 2^k$ *for* $k \geq 0$.

3. $C_n^+ = (n-1)2^{n-2}$ *for* $n \geq 1$, $C_0^+ := 0$.

4. $P_{2k+1}^+ = k2^k$ *for* $k \geq 0$, $P_{2k}^+ = (2k-1)2^{k-1}$ *for* $k \geq 1$, $P_0^+ := 0$.

5. $C_n^s = (n+1)2^{n-2}$, *for* $n \geq 1$, $C_0^s := 1$.

6. $P^s_{2k+1} = (k+1)2^k$ *for* $k \geq 0$, $P^s_{2k} = (2k+1)2^{k-1}$ *for* $k \geq 1$, $P^s_0 := 1$.

Proof: 1. The number of compositions of n into i parts is $\binom{n-1}{i-1}$ (see Section 1.4 in [5]). Thus, for $n \geq 1$,

$$C_n = \sum_{i=1}^{n} \binom{n-1}{i-1} = 2^{n-1}.$$

2. Using the PCM as illustrated in Figure 1, it is easy to see that

$$P_{2k} = P_{2k+1} = \sum_{i=0}^{k} C_i = 1 + (1 + 2 + \cdots + 2^{k-1}) = 2^k.$$

3. A composition of n with i summands has $i-1$ "+" signs. Thus, the number of "+" signs can be obtained by summing according to the number of summands in the composition:

$$\begin{aligned} C_n^+ &= \sum_{i=1}^{n}(i-1)\cdot\binom{n-1}{i-1} = \sum_{i=2}^{n}(i-1)\cdot\frac{(n-1)!}{(i-1)!(n-i)!} \\ &= (n-1)\sum_{i=2}^{n}\binom{n-2}{i-2} = (n-1)\cdot 2^{n-2}. \end{aligned} \tag{1}$$

4. The number of "+" signs in a palindrome of $2k+1$ is twice the number of "+" signs in the associated composition, plus two "+" signs connecting the two compositions with the middle summand.

$$\begin{aligned} P_{2k+1}^+ &= \sum_{i=1}^{k}(2C_i + 2C_i^+) = \sum_{i=1}^{k}(2\cdot 2^{i-1} + 2(i-1)2^{i-2}) \\ &= \sum_{i=1}^{k}(i+1)2^{i-1} = k2^k, \end{aligned}$$

where the last equality is easily proved by induction. For palindromes of $2k$, the same reasoning applies, except that there is only one "+" sign when a composition of k is combined with its mirror image. Thus,

$$\begin{aligned} P_{2k}^+ &= \sum_{i=1}^{k-1}(2C_i + 2C_i^+) + (C_k + 2C_k^+) = \sum_{i=1}^{k}(2C_i + 2C_i^+) - C_k \\ &= k2^k - 2^{k-1} = (2k-1)2^{k-1}. \end{aligned}$$

5. & 6. The number of summands in a composition or palindrome is one more than the number of "+" signs, and the results follows by substituting the previous results into $C_n^S = C_n^+ + C_n$ and $P_n^S = P_n^+ + P_n$. $\square$

Part 4 of Theorem 4 could have been proved similarly to part 1, using the number of palindromes of n into i parts, denoted by P_n^i. These numbers exhibit an interesting pattern which will be proved in Lemma 2.

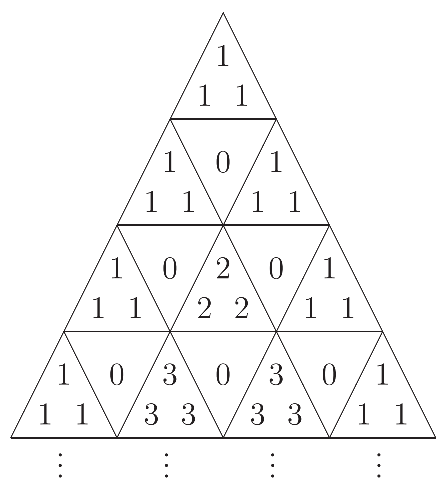

Figure 2: Palindromes with i parts

Lemma 2 $P_{2k-1}^{2j} = 0$ *and* $P_{2k-1}^{2j-1} = P_{2k}^{2j-1} = P_{2k}^{2j} = \binom{k-1}{j-1}$ *for* $j = 1, ..., k, k \geq 1.$

Proof: The first equality follows from the fact that a palindrome of an odd number n has to have an odd number of summands. For the other cases we will interpret the palindrome as a tiling where cuts are placed to create the parts. Since we want to create a palindrome, we look only at one of the two halves of the tiling and finish the other half as the mirror image. If $n = 2k-1$, to create $2j-1$ parts we select $\frac{(2j-1)-1}{2} = j-1$ positions out of the possible $\frac{(2k-1)-1}{2} = k-1$ cutting positions.

If $n = 2k$, then we need to distinguish between palindromes having an odd or even number of summands. If the number of summands is $2j-1$, then there cannot be a cut directly in the middle, so only $\frac{2k-2}{2} = k-1$ cutting positions are available, out of which we select $\frac{(2j-1)-1}{2} = j-1$. If the number of summands is $2j$, then the number of palindromes corresponds to the number of compositions of k, with half the number of summands ($=j$), which equals $\binom{k-1}{j-1}$ $\square$

3. LEVELS, RISES AND DROPS FOR COMPOSITIONS

We now turn our attention to the harder and more interesting results for the numbers of levels, rises and drops in all compositions of n.

Theorem 3 *1.* $l_n = \frac{1}{36}\left((3n+1)2^n + 8(-1)^n\right)$ *for* $n \geq 1$ *and* $l_0 = 0$.

2. $r_n = d_n = \frac{1}{9}\left((3n-5)2^{n-2} - (-1)^n\right)$ *for* $n \geq 3$ *and* $r_0 = r_1 = r_2 = 0$.

Proof: 1. In order to obtain a recursion for the number of levels in the compositions of n, we look at the right end of the compositions, as this is where the CCM creates changes. Applying the CCM, the levels in the compositions of $n+1$ are twice those in the compositions of n, modified by any changes in the number of levels that occur at the right end. If a 1 is added, an additional level is created in all the compositions of n that end in 1, i.e., a total of $C_n(1) = \frac{1}{2}C_{n-1}$ additional levels. If the rightmost summand is increased by 1, one level is lost if the composition of n ends in $x+x$, and one additional level is created if the composition of n ends in $x+(x-1)$. Thus,

$$l_{2k+1} = 2l_{2k} + \frac{1}{2}C_{2k} - \sum_{x=1}^{k} C_{2k}(x,x) + \sum_{x=2}^{k} C_{2k}(x,x-1)$$

$$\begin{aligned}
&= 2l_{2k} + 2^{2k-2} - \sum_{x=1}^{k} C_{2k-2x} + \sum_{x=2}^{k} C_{2k-(2x-1)} \\
&= 2l_{2k} + 2^{2k-2} - (2^{2k-3} + 2^{2k-5} + \cdots + 2^1 + 1) + (2^{2k-4} + \cdots + 1) \\
&= 2l_{2k} + (2^{2k-2} - 2^{2k-3} + 2^{2k-4} - \cdots - 2 + 1) - 1 \\
&= 2l_{2k} + \frac{2^{2k-1} - 2}{3},
\end{aligned}$$

while

$$\begin{aligned}
l_{2k} &= 2l_{2k-1} + \frac{1}{2}C_{2k-1} - \sum_{x=1}^{k-1} C_{2k-1}(x,x) + \sum_{x=2}^{k} C_{2k-1}(x,x-1) \\
&= 2l_{2k-1} + 2^{2k-3} - (2^{2k-4} + 2^{2k-6} + \cdots + 2^2 + 1) + (2^{2k-5} + \cdots + 2^1 + 1) \\
&= 2l_{2k-1} + (2^{2k-3} - 2^{2k-4} + 2^{2k-5} - \cdots + 2 - 1) + 1 \\
&= 2l_{2k-1} + \frac{2^{2k-2} + 2}{3}.
\end{aligned}$$

Altogether, for all $n \geq 2$,

$$l_n = 2l_{n-1} + \tfrac{2^{n-2} + 2(-1)^n}{3}. \tag{2}$$

The homogeneous and particular solutions, $l_n^{(h)}$ and $l_n^{(p)}$, respectively, are given by

$$l_n^{(h)} = c \cdot 2^n \quad \text{and} \quad l_n^{(p)} = A \cdot (-1)^n + B \cdot n2^n.$$

Substituting $l_n^{(p)}$ into Eq. (2) and comparing the coefficients for powers of 2 and -1, respectively, yields $A = \frac{2}{9}$ and $B = \frac{1}{12}$. Substituting $l_n = l_n^{(h)} + l_n^{(p)} = c \cdot 2^n + \frac{2}{9}(-1)^n + \frac{1}{12} \cdot n \cdot 2^n$ into Eq. (2) and using the initial condition $l_2 = 1$ yields $c = \frac{1}{36}$, giving the equation for l_n for $n \geq 3$. (Actually, the formula also holds for $n \geq 1$).

2. It is easy to see that $r_n = d_n$, since for each nonpalindromic composition there is one which has the summands in reverse order. For palindromic compositions, the symmetry matches each rise in the first half with a drop in the second half and vice versa. Since $C_n^+ = r_n + l_n + d_n$, it follows that $r_n = \frac{C_n^+ - l_n}{2}$. □

Table 1 shows values for the quantities of interest. In Theorem 4 we will establish the patterns suggested in this table.

n	1	2	3	4	5	6	7	8	9	10	11	12
C_n^+	0	1	4	12	32	80	192	448	1024	2304	5120	11264
l_n	0	1	2	6	14	34	78	178	398	882	1934	4210
$r_n = d_n$	0	0	1	3	9	23	57	135	313	711	1593	3527

Table 1: Values for C_n^+, l_n and r_n

Theorem 4 *1.* $r_{n+1} = r_n + l_n$ *and more generally,* $r_n = \sum_{i=2}^{n-1} l_i$ *for* $n \geq 3$.

2. $C_n^+ = r_n + r_{n+1}$.

3. $C_n^+ = 4 \cdot (l_{n-1} + l_{n-2}) = 4 \cdot (r_n - r_{n-2})$.

4. $l_n - r_n = a_{n-1}$, *where* a_n *is the* n^{th} *term of the Jacobsthal sequence.*

Proof: 1. The first equation follows by substituting the formulas of Theorem 3 for r_n and l_n and collecting terms. The general formula follows by induction.
2. This follows from part 1, since $C_n^+ = r_n + l_n + d_n$ and $r_n = d_n$.
3. The first equality follows by substituting the formula in Theorem 3 for l_{n-1} and l_{n-2}. The second equality follows from part 1.
4. The sequence of values for $f_n = l_n - r_n$ is given by $1, 1, 3, 5, 11, 21, 43, \ldots$. This sequence satisfies several recurrence relations, for example $f_n = 2f_{n-1} + (-1)^n$ or $f_n = 2^n - f_{n-1}$, both of which can be verified by substituting the formulas given in Theorem 3. These recursions define the Jacobsthal sequence (A001045 in [8]), and comparison of the initial values shows that $f_n = a_{n-1}$. $\square$

4. LEVELS, RISES AND DROPS FOR PALINDROMES

We now look at the numbers of levels, rises and drops for palindromes. Unlike the case for compositions, there is no single formula for the number of levels, rises and drops, respectively. Here we have to distinguish between odd and even values of n, as well as look at the remainder of k when divided by 3.

Theorem 5 *For $k \geq 1$,*

1. $$\tilde{l}_{2k} = \frac{2}{9}(-1)^k + 2^k\left(\frac{53}{126} + \frac{k}{3}\right) + \begin{cases} \frac{6}{7} & k \equiv 0 \ mod\ (3) \\ \frac{-2}{7} & k \equiv 1 \ mod\ (3) \\ \frac{-4}{7} & k \equiv 2 \ mod\ (3) \end{cases}$$
$$\tilde{l}_{2k+1} = \frac{2}{9}(-1)^k + 2^k\left(\frac{22}{63} + \frac{k}{3}\right) + \begin{cases} \frac{-4}{7} & k \equiv 0 \ mod\ (3) \\ \frac{6}{7} & k \equiv 1 \ mod\ (3) \\ \frac{-2}{7} & k \equiv 2 \ mod\ (3) \end{cases}$$

2. $$\tilde{r}_{2k} = \tilde{d}_{2k} = -\frac{1}{9}(-1)^k - 2^{k-1}\left(\frac{58}{63} - \frac{2k}{3}\right) + \begin{cases} \frac{-3}{7} & k \equiv 0 \ mod\ (3) \\ \frac{1}{7} & k \equiv 1 \ mod\ (3) \\ \frac{2}{7} & k \equiv 2 \ mod\ (3) \end{cases}$$
$$\tilde{r}_{2k+1} = \tilde{d}_{2k+1} = -\frac{1}{9}(-1)^k - 2^{k-1}\left(\frac{22}{63} - \frac{2k}{3}\right) + \begin{cases} \frac{2}{7} & k \equiv 0 \ mod\ (3) \\ \frac{-3}{7} & k \equiv 1 \ mod\ (3) \\ \frac{1}{7} & k \equiv 2 \ mod\ (3) \end{cases}$$

Proof: We use the PCM, where a middle summand $m = 2l$ or $m = 2l + 1$ ($l \geq 0$) is combined with a composition of $k - l$ and its mirror image, to create a palindrome of $n = 2k$ or $n = 2k + 1$, respectively. The number of levels in the palindrome is twice the number of levels of the composition, plus any additional levels created when the compositions are joined with the middle summand.

We will first look at the case where n (and thus m) is even. If $l = m = 0$, a composition of k is joined with its mirror image, and we get only one additional level. If $l > 0$, then we get two additional levels for a composition ending in m, for $m = 2l \leq k - l$. Thus,

$$\tilde{l}_{2k} = 2 \cdot \sum_{l=0}^{k} l_{k-l} + C_k + 2 \cdot \sum_{l=1}^{\lfloor k/3 \rfloor} C_{k-l}(2l) = s_1 + 2^{k-1} + s_2. \tag{3}$$

Since $l_0 = l_1 = 0$, the first summand reduces to

$$\begin{aligned} s_1 &= \frac{1}{18} \cdot \sum_{i=2}^{k} \left\{(3i+1)2^i + 8(-1)^i\right\} = \frac{2}{9}\sum_{i=2}^{k} 2^{i-2} + \frac{1}{3}\sum_{i=2}^{k} i \cdot 2^{i-1} + \frac{4}{9}\sum_{i=0}^{k}(-1)^i \\ &= \frac{2}{9} \cdot (2^{k-1} - 1) + \frac{1}{3}\left(\frac{\mathrm{d}}{\mathrm{dx}}\sum_{i=2}^{k} x^i\right)\Bigg|_{x\,=\,2} + \frac{2}{9}((-1)^k + 1) \\ &= \frac{1}{9}2^k + \frac{1}{3}\left\{(k+1)2^k - 2^{k+1}\right\} + \frac{2}{9}(-1)^k = \frac{2}{9}(-1)^k + \left(\frac{k}{3} - \frac{2}{9}\right)2^k. \end{aligned} \tag{4}$$

To compute s_2, note that $C_n(i) = C_{n-1}(i-1) = ... = C_{n-i+1}(1) = \frac{1}{2}C_{n-i+1} = 2^{n-i-1}$ for $i < n$ and $C_n(n) = 1$. The latter case only occurs when $k = 3l$. Let $k := 3j + r$, where $r = 1, 2, 3$. (This somewhat unconventional definition allows for a unified proof.) Thus, with $\mathcal{I}_A$ denoting the indicator function of A,

$$\begin{aligned} s_2 &= 2 \cdot \sum_{l=1}^{\lfloor k/3 \rfloor} C_{k-l}(2l) = 2 \cdot \sum_{l=1}^{j} 2^{3j+r-l-2l-1} + 2 \cdot \mathcal{I}_{\{r=3\}} \\ &= 2^r \cdot \sum_{l=1}^{j} \left(2^3\right)^{j-l} + 2 \cdot \mathcal{I}_{\{r=3\}} = 2^r \left(\frac{(2^3)^j - 1}{7}\right) + 2 \cdot \mathcal{I}_{\{r=3\}} \\ &= \frac{2^k - 2^r}{7} + 2 \cdot \mathcal{I}_{\{r=3\}} = \begin{cases} \frac{2^k+6}{7} & k \equiv 0 \bmod (3) \\ \frac{2^k-2^r}{7} & k \equiv r \bmod (3), \text{ for } r = 1, 2. \end{cases} \end{aligned} \tag{5}$$

Combining Equations (3), (4) and (5) and simplifying gives the result for $\tilde{l}_{2k}$.

For $n = 2k+1$, we make a similar argument. Again, each palindrome has twice the number of levels of the associated composition, and we get two additional levels whenever the composition ends in m, for $m = 2l + 1 \leq k - l$. Thus,

$$\tilde{l}_{2k+1} = 2 \cdot \sum_{l=0}^{k} l_{k-l} + 2 \cdot \sum_{l=0}^{\lfloor (k-1)/3 \rfloor} C_{k-l}(2l+1) =: s_1 + s_3.$$

With an argument similar to that for s_2,we derive

$$s_3 = \begin{cases} \frac{2^{k+2}-4}{7} & k \equiv 0 \bmod (3) \\ \frac{2^{k+2}+6}{7} & k \equiv 1 \bmod (3) \\ \frac{2^{k+2}-2}{7} & k \equiv 2 \bmod (3) \end{cases} \tag{6}$$

Combining Equations (4) and (6) and simplifying gives the result for $\tilde{l}_{2k+1}$. Finally, the results for $\tilde{r}_n$ and $\tilde{d}_n$ follow from the fact that $\tilde{r}_n = \tilde{d}_n = \frac{P_n^+ - \tilde{l}_n}{2}$. □

5. GENERATING FUNCTIONS

Let $G_{a_n}(x) = \sum_{k=0}^{\infty} a_k x^k$ be the generating function of the sequence $\{a_n\}_0^{\infty}$. We will give the generating functions for all the quantities of interest.

Theorem 6 *1.* $G_{C_n}(x) = \frac{1-x}{1-2x}$ *and* $G_{P_n}(x) = \frac{1+x}{1-2x^2}$.

2. $G_{C_n^+}(x) = \frac{x^2}{(1-2x)^2}$ *and* $G_{P_n^+}(x) = \frac{x^2+2x^3+2x^4}{(1-2x^2)^2}$.

3. $G_{C_n^{\mathrm{S}}}(x) = \frac{1-3x+3x^2}{(1-2x)^2}$ *and* $G_{P_n^{\mathrm{S}}}(x) = \frac{1+x-x^2+2x^4}{(1-2x^2)^2}$.

4. $G_{l_n}(x) = \frac{x^2(1-x)}{(1+x)(1-2x)^2}$ *and* $G_{r_n}(x) = G_{d_n}(x) = \frac{x^3}{(1+x)(1-2x)^2}$.

5. $G_{\tilde{l}_n}(x) = \frac{x^2(1+3x+4x^2+x^3-x^4-4x^5-6x^6)}{(1+x^2)(1+x+x^2)(1-2x^2)^2}$ *and*
$G_{\tilde{r}_n}(x) = G_{\tilde{d}_n}(x) = \frac{x^4(1+3x+4x^2+4x^3+4x^4)}{(1+x^2)(1+x+x^2)(1-2x^2)^2}$.

Proof: 1. & 2. The generating functions for $\{C_n\}_0^{\infty}$, $\{P_n\}_0^{\infty}$ and $\{C_n^+\}_0^{\infty}$ are straightforward using the definition and the formulas of Theorem 1. We derive $G_{P_n^+}(x)$, as it needs to take into account the two different formulas for odd and even n. From Theorem 1, we get

$$\begin{aligned} G_{P_n^+}(x) &= \sum_{k=1}^{\infty} P_{2k-1}^+ x^{2k-1} + \sum_{k=1}^{\infty} P_{2k}^+ x^{2k} \\ &= \sum_{k=1}^{\infty} (k-1)2^{k-1} x^{2k-1} + \sum_{k=1}^{\infty} (2k-1)2^{k-1} x^{2k} \end{aligned} \tag{7}$$

Separating each sum in Eq. (7) into terms with and without a factor of k, and recombining like terms across sums leads to

$$\begin{aligned} G_{P_n^+}(x) &= \frac{1+2x}{4} \sum_{k=1}^{\infty} 4xk(2x^2)^{k-1} - (x+x^2) \sum_{k=1}^{\infty} (2x^2)^{k-1} \\ &= \frac{1+2x}{4} \cdot \frac{\mathrm{d}}{\mathrm{dx}} \left(\frac{1}{1-2x^2} \right) - \frac{x+x^2}{1-2x^2} = \frac{x^2+2x^3+2x^4}{(1-2x^2)^2}. \end{aligned}$$

3. Since $C_n^{\mathrm{S}} = C_n + C_n^+$, $G_{C_n^{\mathrm{S}}}(x) = G_{C_n}(x) + G_{C_n^+}(x)$; likewise for $G_{P_n^{\mathrm{S}}}(x)$.

4. The generating function for l_n can be easily computed using Mathematica or Maple, using either the recursive or the explicit description. The relevant Mathematica commands are

```
<<DiscreteMath‘RSolve‘

GeneratingFunction[{a[n+1]==2a[n]+(2/3)*2^(n-2)+(-2/3)*(-1)^(n-2),
   a[0]==0,a[1]==0},a[n],n,z][[1,1]]

PowerSum[((1/36) + (n/12))*2^n + (2/9)*(-1)^n,{z,n,1}]
```

Furthermore, $G_{r_n}(x) = G_{d_n}(x) = \frac{1}{2}\left(G_{C_n^+}(x) - G_{l_n}(x)\right)$, since $r_n = d_n = \frac{C_n^+ - l_n}{2}$.

5. In this case we have six different formulas for $\tilde{l}_n$, depending on the remainder of n with respect to 6. Let $G_i(x)$ denote the generating function of $\left\{\tilde{l}_{6k+i}\right\}_{k=0}^{\infty}$. Then, using the definition of the generating function and separating the sum according to the remainder (similar to the computation in part 2), we get

$$G_{\tilde{l}_n}(x) = G_0(x^6) + x \cdot G_1(x^6) + x^2 \cdot G_2(x^6) + \cdots + x^5 \cdot G_5(x^6).$$

The functions $G_i(x)$ and the resulting generating function $G_{\tilde{l}_n}(x)$ are derived using the following Mathematica commands:

```
<<DiscreteMath‘RSolve‘

g0[z_]=PowerSum[(1/126)((126(n)+53)* 2^(3n)+108+28(-1)^(n)),{z,n,1}]

g1[z_]=PowerSum[(1/63)((63n+22)* 2^(3n)-36+14(-1)^n),{z,n,1}]

g2[z_]=PowerSum[(1/63)((126n+95)* 2^(3n)-18-14(-1)^n),{z,n,0}]

g3[z_]=PowerSum[(1/63)((126n+86)* 2^(3n)+54-14(-1)^n),{z,n,0}]

g4[z_]=PowerSum[(1/63)((252n+274)* 2^(3n)-36+14(-1)^n),{z,n,0}]

g5[z_]=PowerSum[(1/63)((252n+256)* 2^(3n)-18+14(-1)^n),{z,n,0}]

genfun[z]:= g0[z^6]+z g1[z^6]+z^2 g2[z^6]+z^3 g3[z^6]+z^4 g4[z^6]+z^5 g5[z^6]
```

Finally, $G_{\tilde{r}_n}(x) = G_{\tilde{d}_n}(x) = \frac{1}{2}\left(G_{P_n^+}(x) - G_{\tilde{l}_n}(x)\right)$, since $\tilde{r}_n = \tilde{d}_n = \frac{P_n^+ - \tilde{l}_n}{2}$. $\square$

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