THE GAME CREATION OPERATOR

Joint work with Urban Larsson and Matthieu Dufour

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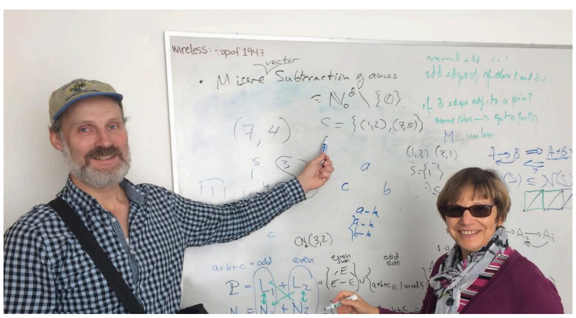

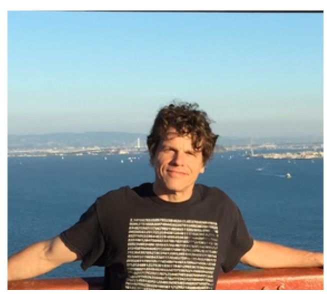

Much of this work was done during my sabbatical visit to Berkeley in October 2015

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Vector Subtraction Games

- A **vector subtraction** or **vector take-away game** is played on one or more stacks of tokens
- **Positions** are described as vectors of stack heights
- The **subtraction set M** consists of the possible moves in the form of subtraction vectors. A move can be used as long as it does not result in negative stack height(s)

(5, 3, 2, 1)

Take one token from stack 1

$\leftrightarrow$

(1, 0, 0, 0)

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Misère-Play ★-Operator

Observation: For vector subtraction games, positions and allowed moves have the **same structure**! This allows us to iteratively create new games.

The **misère ★-operator** is defined as follows:

- We start with a subtraction game **M** that is described by the allowed moves.
- We compute the set of P-positions, **P(M)**
- The losing positions of **M** become the moves for the game **M**★

- Notation: $\mathbf{M}^0 = \mathbf{M}$, $\mathbf{M}^n = (\mathbf{M}^{n-1})^\star$
- **M** is **reflexive** if **M** = **M**★

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Complementary Beatty Sequences & Games

Duchêne-Rigo Conjecture: Every complementary pair of homogeneous Beatty sequences forms the set of losing positions for some invariant impartial game.

This conjecture was proved by Larsson, Hegarty and Fraenkel using the normal-play ★-operator

$$\mathbf{M}^{\star} = P(\mathbf{M}) - \mathbf{0}; \ \mathbf{M} \text{ is reflexive if } \mathbf{M} = \mathbf{M}^{\star\star}$$

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Questions for Misère-Play ★-Operator

- **Question 1**: Does the misère-play ★-operator converge (point-wise)?
- **Question 2**: What feature(s) of **M** determine(s) the limit game for its sequence?
- **Question 3**: Limit games are (by definition) reflexive. What is the structure of reflexive games and/or limit games (if they exist)?
- **Question 4**: How quickly does convergence occur?

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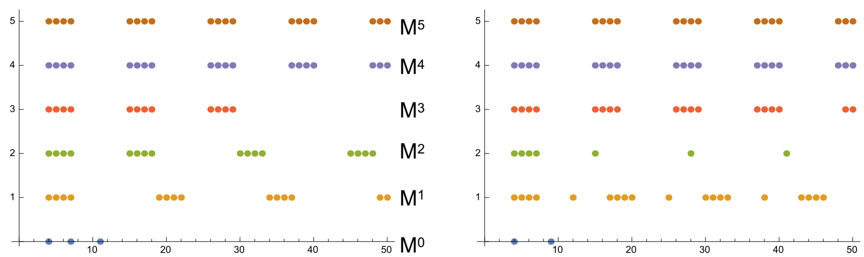
Example for **One** Stack

Misère-Play ★-operator applied five times to initial game



$M^0 = \{4, 7, 11\}$ $G^0 = \{4, 9\}$

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Observations from Example

- Looks like there is convergence (fixed point) for each of the two games
- Limit games seem to have a periodic structure: blocks of moves alternate with blocks of non-moves
- $M^0 = \{4, 7, 11\}$ and $G^0 = \{4, 9\}$ seem to have the same limit game

Question: What do the two sets M^0 and G^0 have in common?

Answer: The minimal element, **k = 4**.

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Q1: Convergence Result

Theorem
Starting from any game **M** on *d* stacks, the sequence of games created by the misère-play ★-operator converges to a (reflexive) limit game $\mathbf{M}^{\infty}$.

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Convergence Result

Proof idea: (for *d* stacks)

- Weight of a position = sum of stack heights = number of tokens
- Inductive proof:
  - For game M^n, all positions of weight n or less are fixed as either move or non-move.
  - Basis: In game M^0, the position of weight 0 (the zero vector) is fixed as an N-position
- Convergence is faster than increase by one token

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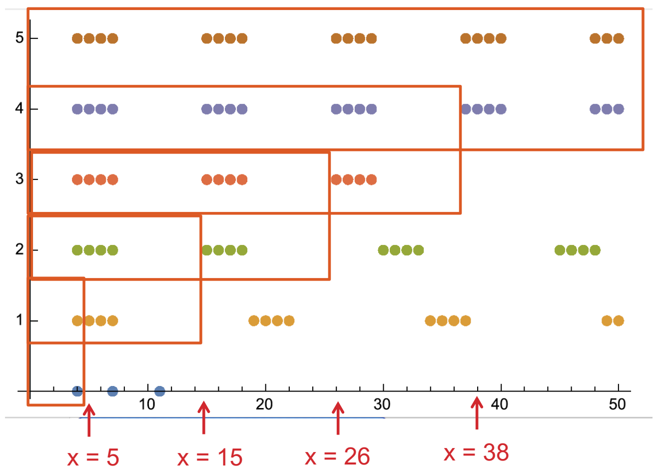
Example



$M^0 = \{4, 7, 11\}$

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Q2: Which Feature of **M** Determines **M**$^\infty$?

Theorem

Two games **M** and **G** (played on the same number of stacks) have the same limit game if and only if their unique **sets of minimal elements** (with the usual partial order) are the same.

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Q3: Characteristic of Reflexive Games

The following lemma is used to prove specific results for one and two stacks.

Lemma
The game **A** on *d* stacks is reflexive if and only if its set of moves **A** (as a set) satisfies

$$\mathbf{A} + \mathbf{A} = \mathbf{A}^c \setminus T_{\mathbf{A}}$$

where $T_{\mathbf{A}}$ is the set of terminal positions of the game **A**.

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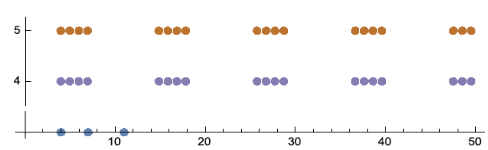
Structure of Reflexive Games on One Stack



Pattern:
- period 3k-1;
- starts at k
- has k moves, followed by 2k-1 non-moves.

$\mathbf{M}_k := \{ i\, p_k + k, \ldots, i\, p_k + (2k - 1) \mid i = 0,1,\ldots\}$, where $p_k = 3k-1$

Theorem
The game **M** is reflexive iff $\mathbf{M} = \mathbf{M}_k$ for some $k > 0$.

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Structure of Limit/Reflexive Games on Two Stacks

Classification of games according to minimal moves

① Exactly one minimal move
  a. Not on an axis
  b. On one of the axes

② Exactly two minimal moves
  a. No minimal move on an axis
  b. Exactly one move is on an axis
  c. **Both moves are on the axes**

③ Three or more minimal moves
  a. **No minimal move on an axis**
  b. Exactly one move is on an axis
  c. **Two moves are on the axes**

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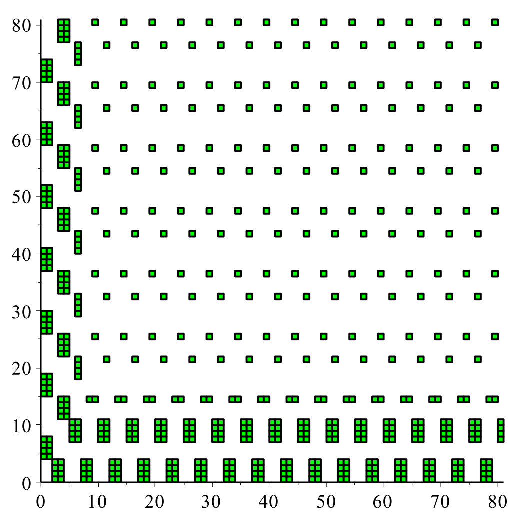
Example: Two Minima on Axes



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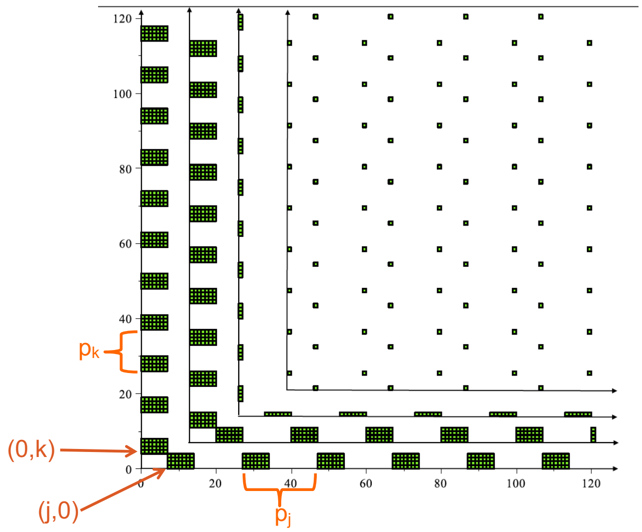
Definition of Game $M_{j,k}$



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Reflexivity of $M_{j,k}$

Theorem [Bloomfield, Dufour, Heubach, Larsson]
The game $\mathbf{M_{j,k}}$ is reflexive.

Corollary
The limit game of a set **M** equals the game $\mathbf{M_{j,k}}$ if and only if the **set of minimal elements** of **M** is {**(j,0),(0,k)**}.

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Q4: How Long until Convergence?

- We can only answer this question for games on **one** stack and for specific initial games

Theorem
For **M = {*k*}** with k > 1 it takes exactly **5** iterations for the limit game to appear for the first time.

Proof: We explicitly derive the games M^1 through M^5.

- For games on two stacks we have very varied results from our computer explorations

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Future Work

1. Investigate the structure of the limit games in the other classes for games on two stacks

   - Computer experiments for three minimal elements have produced “L-shaped” limit games, limit games with diagonal stripes, and limit games that combine the two features

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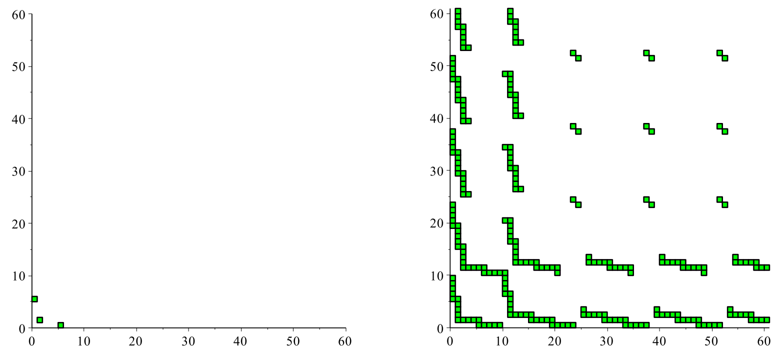
Three Minimal Moves – Two on Axes

M = {(0,5), (1,1), (5,0)}



Convergence after 8 steps

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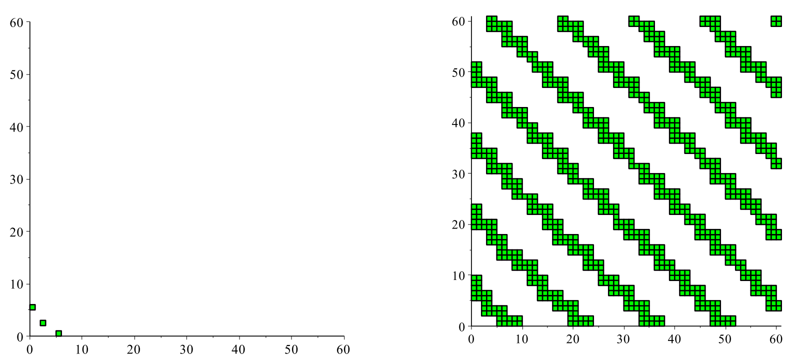
Three minimal moves – two on axes

M = {(0,5), (2,2), (5,0)}



Convergence after 7 steps

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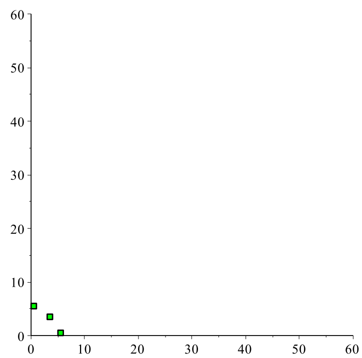
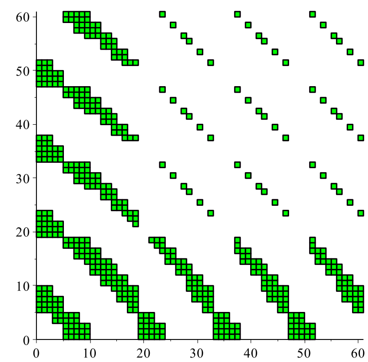
Three Minimal Moves – Two on Axes

M = {(0,5), (3,3), (5,0)}





Convergence after 7 steps

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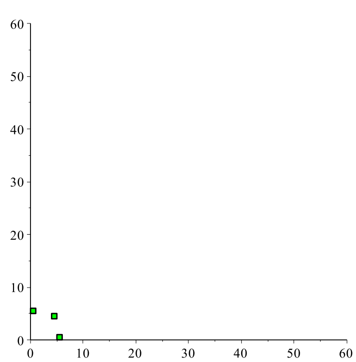
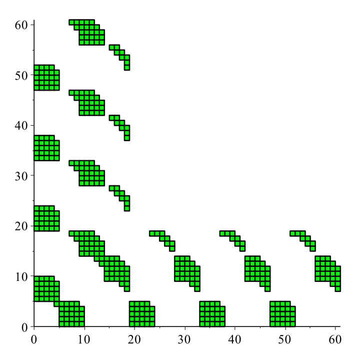
Three Minimal Moves – Two on Axes

M = {(0,5), (4,4), (5,0)}





Convergence after 6 steps

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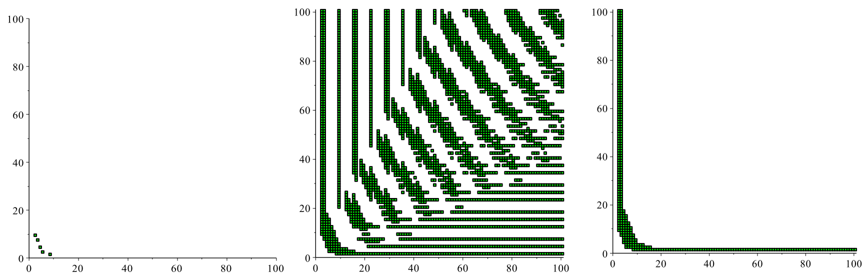
More than three Minimal Moves – None on Axis

M = {(2,9), (3,7), (4,4), (5,2), (8,1)}



Convergence after 2 steps

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Future Work

1. Investigate the structure of the limit games in the other classes for games on two stacks

1. Number of steps to convergence, or showing that it happens in a finite number of steps for all games or for games of a particular (sub-) class

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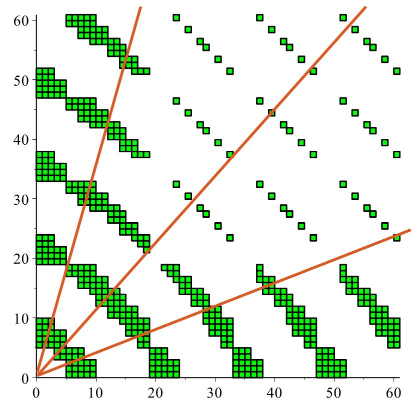
Future Work

Conjecture
For all vector subtraction games on two stacks, limit games under the misère-play ★-operator are ultimately periodic along any line of rational slope.



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- U. Larsson, P. Hegarty, and A. S. Fraenkel. Invariant and dual subtraction games resolving the Duchêne-Rigo conjecture, *Theoretical Computer Science*, 412, pp 729-735, 2011.
- U. Larsson. The ★-operator and invariant subtraction games. *Theoretical Computer Science*, 422, pp 52-58, 2012.
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THANK YOU!

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Slides will be posted on my web site

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