HANDOUT SOLUTIONS

(1) Note that $[F(\alpha,\beta):F] = [F(\alpha,\beta):F(\alpha)][F(\alpha):F]$ and $[F(\alpha,\beta):F] = [F(\alpha,\beta):F(\beta)][F(\beta):F]$. So, m and n both divide $[F(\alpha,\beta):F]$.

Thus, $[F(\alpha,\beta):F] = mk$ for some integer k. Hence n divides mk. Since $\gcd(n,m)=1$, we have that n divides k. Thus, $[F(\alpha,\beta):F] = mn\ell$ where $\ell \in \mathbb{Z}$.

Note that $[F(\alpha,\beta):F(\alpha)] = \left(\text{degree of } \min_{\beta,F(\alpha)}(x)\right)$ and $[F(\beta):F] = \left(\text{degree of } \min_{\beta,F}(x)\right)$. And degree of $\min_{\beta,F(\alpha)}(x) \leq$ degree of $\min_{\beta,F}(x)$. Hence,

$$[F(\alpha,\beta):F] = [F(\alpha,\beta):F(\alpha)][F(\alpha):F]$$
$$\leq [F(\beta):F][F(\alpha):F] = mn.$$

So, $[F(\alpha,\beta):F] = mn$.

(2) Note that $\min_{2^{1/3},\mathbb{Q}}(x) = x^3 - 2$. Hence

$[\mathbb{Q}(2^{1/3}) : \mathbb{Q}] = 3$. ($x^3-2$ is irreducible by Eisenstein with $p=2$)

Note that $i \notin \mathbb{Q}(2^{1/3})$ since

$\mathbb{Q}(2^{1/3}) = \{a + b2^{1/3} + c2^{2/3} \mid a,b,c \in \mathbb{Q}\} \subseteq \mathbb{R}$

and $i \notin \mathbb{R}$. Hence x^2+1 is irreducible over $\mathbb{Q}(2^{1/3})$. So, $\min_{i,\mathbb{Q}(2^{1/3})}(x) = x^2+1$.

So,

$$[\mathbb{Q}(2^{1/3}, i) : \mathbb{Q}] = [\mathbb{Q}(2^{1/3}, i) : \mathbb{Q}(2^{1/3})][\mathbb{Q}(2^{1/3}) : \mathbb{Q}]$$

$$= 2 \cdot 3 = 6.$$

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(3) Do the same thing as #2. Try it.

Change α to $(1+3^{1/4})^{1/3}$. Typo on handout.

(4) Let $\alpha = (1+3^{1/4})^{1/3}$. Then

$\alpha^3 = 1 + 3^{1/4}$. So, $\alpha^3 - 1 = 3^{1/4}$.

Thus, $(\alpha^3 - 1)^4 = 3$. So,

$\alpha^{12} - 4\alpha^9 + 6\alpha^6 - 4\alpha^3 - 2 = 0$.

Let $p(x) = x^{12} - 4x^9 + 6x^6 - 4x^3 - 2$.

Then $p(\alpha) = 0$ and $p(x)$ is

irreducible using Eisenstein's criteria

and $q = 2$.