(2) Suppose that $|G| = pq$ where p, q are primes and that G is abelian. By Cauchy's thm there exist $x, y \in G$ with $|x| = p$ and $|y| = q$.

<u>Claim:</u> $|xy| = pq$.

Note that

$$(xy)^{pq} \underset{\uparrow}{=} x^{pq} y^{pq} = (x^p)^q (y^q)^p = 1^q \cdot 1^p = 1.$$

$\boxed{\text{because } G \text{ is abelian}}$

Thus, $1 \leq |xy| \leq pq$.

Let $H = \langle x \rangle$ and $K = \langle y \rangle$. Then $|H| = p$ and $|K| = q$. Since $H \cap K \leq H$ and $H \cap K \leq K$, by Lagranges theorem, $|H \cap K|$ divides p and $|H \cap K|$ divides q. Since q, p are primes and $p \neq q$ we know that $|H \cap K| = 1$, Thus, $H \cap K = \{1\}$.

Now suppose that $(xy)^k = 1$ for some $k > 1$. Then $x^k y^k = 1$. So,

$x^k = y^{-k}$. Thus, $x^k \in K$ and $y^{-k} \in H$. So, $x^k = 1$ and $y^{-k} = 1$. Hence $x^k = 1$ and $y^k = 1$. Thus, p divides k and q divides k. Since p and q are primes we have that k is a multiple of pq. Hence $k \geq pq$. Thus, $|xy| = pq$.

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(15) $\mathbb{Z}_5^\times = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$$\left.\begin{array}{l} \bar{2}^1 = \bar{2} \\ \bar{2}^2 = \bar{4} \\ \bar{2}^3 = \bar{8} = \bar{3} \\ \bar{2}^4 = \bar{16} = \bar{1} \end{array}\right\} \text{Hence } \mathbb{Z}_5^\times = \langle \bar{2} \rangle.$$

$\mathbb{Z}_9^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$

$\langle \bar{2} \rangle = \{\bar{2}, \bar{4}, \bar{8}, \overline{16} = \bar{7}, \overline{14} = \bar{5}, \overline{10} = \bar{1}\} = \mathbb{Z}_9^\times$

You do $\mathbb{Z}_{18}^\times$.