7.4

(8) Note that if $a=0$, then $(a)=0$, So, $(a)=(b)$ iff $b=0$ iff $a=0=b\cdot 1$. Now suppose $a\neq 0, b\neq 0$.

Suppose $(a)=(b)$. Then $a\in(b)$ and $b\in(a)$.

So, $a=bx$ where $x\in R$ and $b=ay$ where $y\in R$. So, $a=bx=ayx$.

Thus, $a(1-yx)=0$. Since we are assuming that $a\neq 0$ we must have $1-yx=0$ since R is an integral domain so, $yx=1$. So, $a=bx$ where x is a unit.

Now suppose $a=ub$ where u is a unit.

Let $x\in(a)$. Then $x=ar$ where $r\in R$.

So, $x=ar=ubr=b(ur)\in(b)$.

So, $(a)\subseteq(b)$.

Since u is a unit there exists $u^{-1}\in R$ with $uu^{-1}=u^{-1}u=1$. So, $u^{-1}a=b$.

Let $y\in(b)$. Then, $y=bs$ where $s\in R$.

So, $y=bs=u^{-1}as=u^{-1}sa\in(a)$.

So, $(b)\subseteq(a)$. Thus, $(a)=(b)$.

(9) $I = \{ f : [0,1] \to [0,1] \mid f(\frac{1}{3}) = f(\frac{1}{2}) = 0 \}$.

The zero function is in I.

Also, if $f, g \in I$, then

$$(f-g)(\tfrac{1}{3}) = f(\tfrac{1}{3}) - g(\tfrac{1}{3}) = 0 - 0 = 0$$

and $$(f-g)(\tfrac{1}{2}) = f(\tfrac{1}{2}) - g(\tfrac{1}{2}) = 0 - 0 = 0.$$

So, $f - g \in I$.

So, I is a subgroup of R.

Let $h \in R$. and $f \in I$.

Then, $(hf)(\frac{1}{3}) = h(\frac{1}{3})f(\frac{1}{3}) = h(\frac{1}{3}) \cdot 0 = 0$

and $(hf)(\frac{1}{2}) = h(\frac{1}{2})f(\frac{1}{2}) = h(\frac{1}{2}) \cdot 0 = 0$

So, $hf \in I$.

Similarly $fh \in I$.

So, I is an ideal of R.

Let f be a function in R with $f(\frac{1}{3}) = 0$ and $f(\frac{1}{2}) \neq 0$. Let h be a function in R with $h(\frac{1}{3}) \neq 0$ and $h(\frac{1}{2}) = 0$.

Then $fh \in I$ but $f \notin I$ and $h \notin I$,

So, I is not a prime ideal.

(10) Let $x, y \in R$ with $xy = 0$.
Then $xy \in P$ since $0 \in P$. Since P is prime, either $x \in P$ or $y \in P$.
But P has no zero divisors, so either $x = 0$ or $y = 0$.
So, R is an integral domain.

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(14) I'm going to do this problem when $R=F$ is a field.

(a) Let $f(x)$ be a polynomial of degree n.
Let $p(x) \in F[x]$. Then $p(x) = q(x) f(x) + r(x)$ where $q(x), r(x) \in F[x]$ and $r(x) = 0$ or $\text{degree}(r) < \text{degree}(f) = n$. ~~Because~~ This gives
$$p(x) + (f(x)) = r(x) + (f(x)).$$

(b) Suppose $p(x) + (f(x)) = q(x) + (f(x))$.
Then, $p(x) - q(x) = f(x) g(x)$ for some $g(x) \in F[x]$.
So, $\text{degree}(p-q) = \text{degree}(fg) \geq n$ which can't happen.

Let $I = (x^2 + x + \bar{1})$.

15 (a) By 14(a)

$$\bar{E} = \mathbb{Z}_2[x]/(x^2+x+\bar{1}) = \{\bar{a}x + \bar{b} + I \mid \bar{a}, \bar{b} \in \mathbb{Z}_2\}$$

$$= \{\bar{0} + I, \bar{1} + I, x + I, \bar{1} + x + I\}.$$

Note that

$$x^2 + I = -(x + \bar{1}) + I = (x + \bar{1}) + I.$$

(b)

$(\bar{E}, +)$	$\bar{0}+I$	$\bar{1}+I$	$x+I$	$(\bar{1}+x)+I$
$\bar{0}+I$	$\bar{0}+I$	$\bar{1}+I$	$x+I$	$(\bar{1}+x)+I$
$\bar{1}+I$	$\bar{1}+I$	$\bar{0}+I$	$\bar{1}+x+I$	$x+I$
$x+I$	$x+I$	$\bar{1}+x+I$	$\bar{0}+I$	$\bar{1}+I$
$(\bar{1}+x)+I$	$(1+x)+I$	$x+I$	$\bar{1}+I$	$\bar{0}+I$

since each element has order 2, $E \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

(c)

$(\bar{E}^{*}, \cdot)$	$\bar{1}+I$	$x+I$	$(\bar{1}+x)+I$
$\bar{1}+I$	$\bar{1}+I$	$x+I$	$(\bar{1}+x)+I$
$x+I$	$x+I$	$\bar{1}+x+I$	$\bar{1}+I$
$(\bar{1}+x)+I$	$(\bar{1}+x)+I$	$\bar{1}+I$	$x+I$

ex:

$$[(\bar{1}+x)+I][(\bar{1}+x)+I]$$
$$= \bar{1} + \bar{2}x + x^2 + I$$
$$= \bar{1} + \bar{1} + x + I = x + I$$