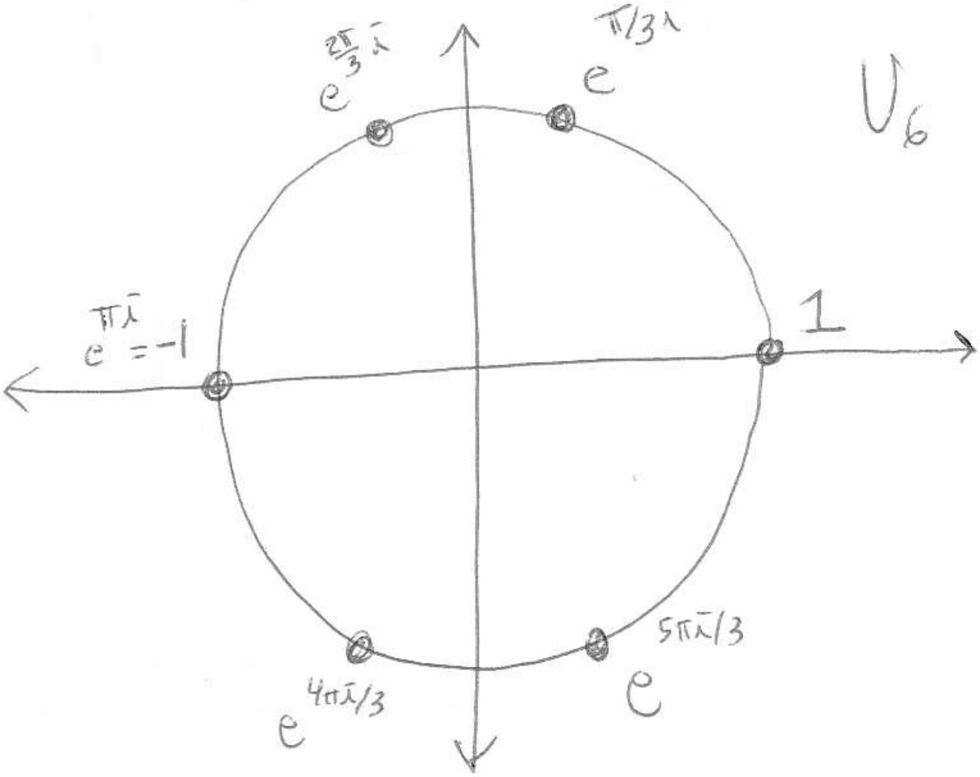
Homework #1

(1) I'll do $\mathbb{Z}_3$, you do $\mathbb{Z}_6$.

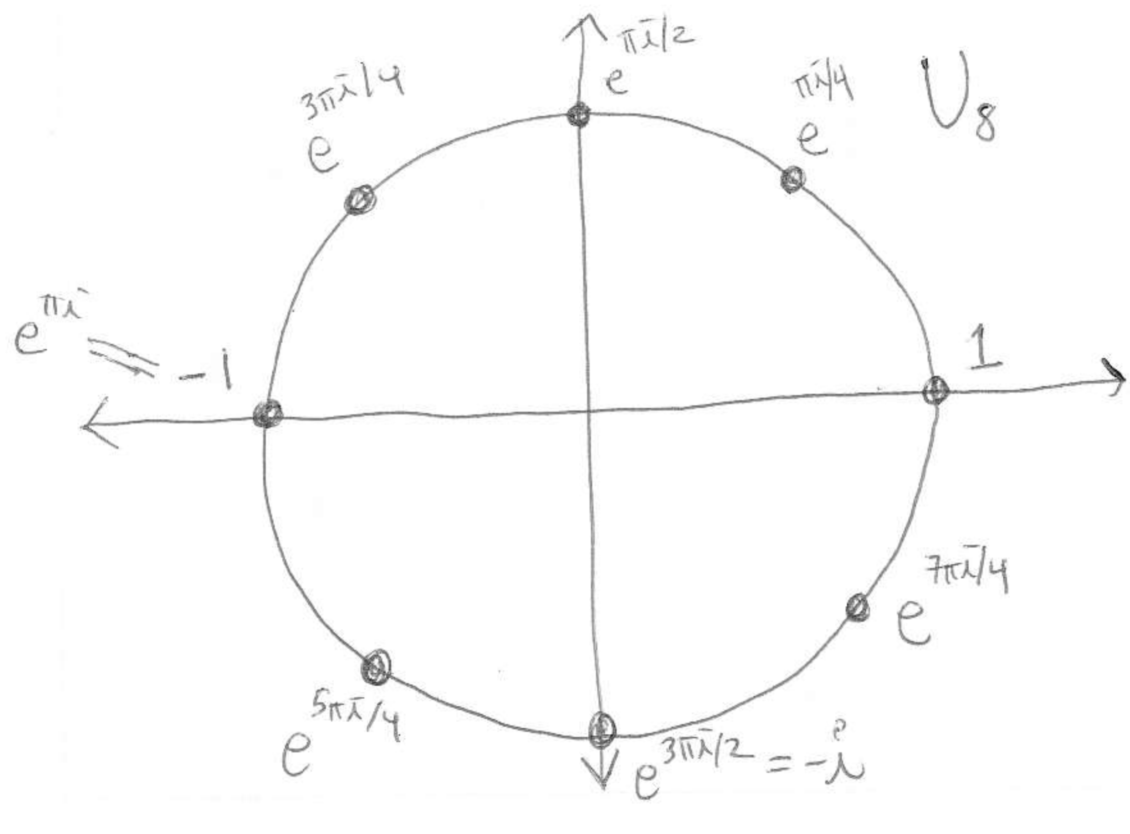
$\mathbb{Z}_3, +$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

$\mathbb{Z}_3, \cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{1}$

(2) $\frac{2\pi}{6} = \frac{\pi}{3} \rightsquigarrow 60^\circ$



$\frac{2\pi}{8} = \frac{\pi}{4} \rightsquigarrow 45^\circ$



③ $U_4 = \{1, e^{\pi i/2}, e^{\pi i}, e^{3\pi i/2}\}$

$= \{1, i, -1, -i\}$

$U_4, \circ$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

$\mathbb{Z}_4, +$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

The tables are the same if you swap the following

$1 \longleftrightarrow \bar{0}$

$i \longleftrightarrow \bar{1}$

$-1 \longleftrightarrow \bar{2}$

$-i \longleftrightarrow \bar{3}$

try it.

(4) $3\mathbb{Z}$ is a group under addition.

Proof:

(closure) Let $x, y \in 3\mathbb{Z}$. Then $x = 3k$ and $y = 3l$ for some $k, l \in \mathbb{Z}$. Then,
$x + y = 3(k+l) \in 3\mathbb{Z}$.

(associativity) $3\mathbb{Z} \subseteq \mathbb{Z}$ and $\mathbb{Z}$ is associative under $+$. So, $3\mathbb{Z}$ is too.

(identity) $0 = 3(0) \in 3\mathbb{Z}$. And

$$0 + 3k = 3k + 0 = 3k \quad \text{for all } k \in \mathbb{Z}.$$

(inverse) Let $x \in 3\mathbb{Z}$. Then, $x = 3k$ for some $k \in \mathbb{Z}$. Note that $3(-k) \in 3\mathbb{Z}$

and

$$3k + 3(-k) = 0 = 3(-k) + 3(k).$$

So, $3(-k)$ is the inverse of $3k$.

(5) There is no identity element.

Suppose $e * a = a * e = a$ for all $a \in \mathbb{R}^+$. Then

$$\sqrt{ea} = \sqrt{ae} = a$$

for all $a \in \mathbb{R}^+$.

For example,

$$\sqrt{2e} = 2 \quad (a=2)$$

and

$$\sqrt{e} = 1 \quad (a=1).$$

Thus,

$$2e = \sqrt{2} \quad \text{and} \quad e = 1.$$

But you can't have both $e = \frac{\sqrt{2}}{2}$ and $e=1$.

Hence, there is no such e.

(6) Associativity fails.

For example,

$$1 * (2 * 3) = 1/(2*3) = 1/(2/3) = 3/2$$

and

$$(1 * 2) * 3 = (1*2)/3 = (1/2)/3 = 1/6$$

and $3/2 \neq 1/6$.

⑦

(closure) Let $x, y \in \mathbb{R}\setminus\{-1\}$. Then, $x \neq -1$ and $y \neq -1$.

Note that $x * y = x + y + xy$.

So, $x * y \in \mathbb{R}$. Is, $x * y = -1$? Suppose that $x + y + xy = -1$. Then, $x(1+y) = -1-y$.

So, $x = \frac{-1-y}{1+y} = -1$. But $x \neq -1$. So, $x * y \neq -1$.

Thus, $x * y \in \mathbb{R}\setminus\{-1\}$.

(associativity) Let $x, y, z \in \mathbb{R}\setminus\{-1\}$.

Then,

$$\begin{aligned} x * (y * z) &= x + (y * z) + x(y * z) \\ &= x + (y + z + yz) + x(y + z + yz) \\ &= x + y + z + yz + xy + xz + xyz \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x * y) + z + (x * y)z \\ &= (x + y + xy) + z + (x + y + xy)z \\ &= x + y + xy + z + xz + yz + xyz \end{aligned}$$

Note that $x * (y * z) = (x * y) * z$.

(identity) Note that $0 \in \mathbb{R} \setminus \{-1\}$. Let $x \in \mathbb{R} \setminus \{-1\}$. Then

$$0 * x = 0 + x + 0x = x$$

and $$x * 0 = x + 0 + x0 = x.$$

Thus, 0 is an identity for $\mathbb{R} \setminus \{-1\}$ under $*$.

(inverses) Let $x \in \mathbb{R} \setminus \{-1\}$. Let $y = \frac{-x}{1+x}$. Note that $y \neq -1$.

[Why? If $-1 = \frac{-x}{1+x}$, then $-1-x = -x$. That would imply that $-1 = 0$ which isn't true.]

Also,

$$x * y = x + \frac{-x}{1+x} + x\left(\frac{-x}{1+x}\right) = \frac{x(1+x)}{1+x} + \frac{-x}{1+x} - \frac{x^2}{1+x}$$

$$= \frac{x + x^2 - x - x^2}{1+x} = 0$$

and $$y * x = \frac{-x}{1+x} + x + \left(\frac{-x}{1+x}\right)x = \frac{-x + x(1+x) - x^2}{1+x}$$

$$= 0$$

So, the inverse of x is $\frac{-x}{1+x} \in \mathbb{R} \setminus \{-1\}$.

(8) Let e be the identity element of G. Note that $e * e = e$, so e is an idempotent element of G.

Suppose $x \in G$ and $x * x = x$.

Then,

$$x^{-1} * (x * x) = x^{-1} * x.$$

So,

$$e * x = e.$$

So, $x = e$.

Thus, e is the only idempotent element of G.

(9) Suppose e is the identity element of a group G where $x * x = e$ for all $x \in G$.

Let $a, b \in G$. Then,

$$(a*b)*(a*b) = e.$$

So, $a*b*a*b = e$.

Note that $a*a = e$ and $b*b = e$ by our assumption on G.

Hence,

$$\underbrace{a*(a}_{e}*b*a*b) = a*e$$

gives

$$b*a*b = a.$$

So,

$$\underbrace{b*b}_{e}*a*b = b*a$$

So, $a*b = b*a$.

So, G is abelian.

~~5. [5 points] Use the method of iteration to find a formula expressing s_n as a function of n where~~

~~$$s_n = 7s_{n-1} + 7^n \quad \text{and} \quad s_0 = 7.$$~~

10 Let $a, b \in G$.

Note that $(a * b)^1 = a * b$.

Let $k \geq 1$. Assume that $(a * b)^k = (a^k) * (b^k)$.

Then,

$$(a * b)^{k+1} = (a * b)^k * (a * b) = (a^k) * (b^k) * a * b$$

$$= (a^k * a) * (b^k * b) = (a^{k+1}) * (b^{k+1}).$$

Since G is abelian

So, by induction, $(a * b)^n = (a^n) * (b^n)$

for all $n \geq 1$.

D_6	1	r	r^2	s	sr	sr^2
1	1	r	r^2	s	sr	sr^2
r	r	r^2	1	sr^2	s	sr
r^2	r^2	1	r	sr	sr^2	s
s	s	sr	sr^2	1	r	r^2
sr	sr	sr^2	s	r^2	1	r
sr^2	sr^2	s	sr	r	r^2	1

D_8	1	r	r^2	r^3	s	sr	sr^2	sr^3
1	1	r	r^2	r^3	s	sr	sr^2	sr^3
r	r	r^2	r^3	1	sr^3	s	sr	sr^2
r^2	r^2	r^3	1	r	sr^2	sr^3	s	sr
r^3	r^3	1	r	r^2	sr	sr^2	sr^3	s
s	s	sr	sr^2	sr^3	1	r	r^2	r^3
sr	sr	sr^2	sr^3	s	r^3	1	r	r^2
sr^2	sr^2	sr^3	s	sr	r^2	r^3	1	r
sr^3	sr^3	s	sr	sr^2	r	r^2	r^3	1

(12)

In D_6: $1^{-1}=1$, $r^{-1}=r^2$, $(r^2)^{-1}=r$,

$s^{-1}=s$, $(sr)^{-1}=sr$, $(sr^2)^{-1}=sr^2$

In D_8: $r^{-1}=r^3$, $(r^2)^{-1}=r^2$,

$(sr)^{-1}=sr$, $(sr^2)^{-1}=sr^2$

In D_{2n}, $r\cdot r^{n-1}=r^n=1$. So, $r^{-1}=r^{n-1}$.

In D_{2n}, $(sr^{i})(sr^{i})=ssr^{-i}r^{i}=s^2\cdot 1=1$

So, $(sr^{i})^{-1}=sr^{i}$

(13) In U_6:

$$1^{-1} = 1, \quad \left(e^{\pi/3 i}\right)^{-1} = e^{5\pi i/3}, \quad \left(e^{2\pi/3 i}\right)^{-1} = e^{4\pi i/3},$$

$$(-1)^{-1} = -1, \quad \left(e^{4\pi i/3}\right)^{-1} = e^{2\pi i/3}, \quad \left(e^{5\pi i/3}\right)^{-1} = e^{\pi i/3}$$

In U_8:

Inverses are

$$1 \longleftrightarrow 1$$

$$e^{\pi i/4} \longleftrightarrow e^{7\pi i/4}$$

$$e^{\pi i/2} \longleftrightarrow e^{3\pi i/2}$$

$$e^{3\pi i/4} \longleftrightarrow e^{5\pi i/4}$$

$$-1 \longleftrightarrow -1$$

(14) In $\mathbb{Z}_6$:

$\overline{0} + \overline{0} = \overline{0} \longrightarrow$ So, $\overline{0}$ is its own inverse.

$\overline{1} + \overline{5} = \overline{0} \longrightarrow$ So, $\overline{1}$ and $\overline{5}$ are inverses.

$\overline{2} + \overline{4} = \overline{0} \longrightarrow$ So, $\overline{2}$ and $\overline{4}$ are inverses.

$\overline{3} + \overline{3} = \overline{0} \longrightarrow$ So, $\overline{3}$ is its own inverse.