Math 465 - Homework # 1
Supremum / Infimum and Absolute value

1. For each of the following subsets of $\mathbb{R}$ find the supremum and infimum if they exist.

   (a) $X = \{5 + 1/n \mid n \in \mathbb{N}\}$

   (b) $X = \{1 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$

   (c) $X = \{\frac{1}{1+x^2} \mid x \in \mathbb{R}\}$

   (d) $X = \{\frac{x}{1+x} \mid x \in \mathbb{R} \text{ with } x > -1\}$

   (e) $X = \{x \in \mathbb{R} \mid x^2 + x < 3\}$

   (f) $X = \{x \in \mathbb{R} \mid x^3 < 1\}$

   (g) $X = \{.3, .33, .333, .3333, .33333, \ldots\}$

2. Let $x \geq 0$ be a real number. Suppose that for each $\epsilon > 0$ we have that $x \leq \epsilon$. Prove that $x = 0$.

3. Suppose that S is a non-empty subset of the real numbers. Suppose that the supremum of S exists. Prove that it is unique. (A similar proof will work to show that infimums are unique.)

4. Let S be a non-empty subset of the real numbers. Suppose that b is an upper bound for S and $b \in S$. Prove that b is the supremum of S.

5. Let A and B be non-empty subsets of $\mathbb{R}$. Suppose that the supremum of A and supremum of B exist. Are the following true or false? If true, prove it. If false, give a counterexample.

   (a) If $A \cap B$ is non-empty then $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$

   (b) If $A \cap B$ is non-empty then $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$

   (c) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

6. Suppose that A and B are non-empty bounded subsets of $\mathbb{R}$. Further suppose that $A \subseteq B$.

   (a) Prove that $\sup(A) \leq \sup(B)$ and $\inf(A) \geq \inf(B)$.

(b) If $\sup(A) = \sup(B)$ and $\inf(A) = \inf(B)$ must it be that $A = B$? If so, prove it. If not, give a counterexample.

7. Let a, b, x, and y be real numbers. Prove:

   (a) If $a < x < b$ and $a < y < b$, then $|x - y| < b - a$.

   (b) $|a - b| = |b - a|$

   (c) $|ab| = |a|\ |b|$

   (d) $|\ |a| - |b|\ | \leq |a - b|$