Hw #2

① $H = \{1, s, sr, sr^2\}$

$G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

H	1	s	sr	sr^2
1	1	s	sr	sr^2
s	s	1	r	r^2
sr	sr	r^2	1	r
sr^2	sr^2	r	r^2	1

H is not a subgroup of D_6 because it is not closed. For example, $s(sr) = r \notin H$.

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$\boxed{\text{In } \mathbb{Z}_5}$

element	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
order	1	5	5	5	5

$\boxed{\text{In } \mathbb{Z}_8}$

element	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
order	1	8	4	8	2	8	4	8

$\boxed{\text{In } U_3}$ $\rho = e^{2\pi i/3}$

element	1	ρ	ρ^2
order	1	3	3

$\boxed{\text{In } D_6}$

element	1	r	r^2	s	sr	sr^2
order	1	3	3	2	2	2

③ $4\mathbb{Z} \leq \mathbb{Z}$

proof;

closure; Let $x, y \in 4\mathbb{Z}$. Then, $x = 4k$ and $y = 4l$ for some $k, l \in \mathbb{Z}$. Then $x + y = 4(k+l) \in 4\mathbb{Z}$ since $k + l \in \mathbb{Z}$.

identity; $0 = 4(0) \in 4\mathbb{Z}$.

inverses; Let $x \in 4\mathbb{Z}$. Then $x = 4k$ where $k \in \mathbb{Z}$. Note that $-x = 4(-k) \in 4\mathbb{Z}$ and

$$x + (-x) = 0 = (-x) + x.$$

So the inverse of x is in $4\mathbb{Z}$.

④ $H = \{2^n \mid n \in \mathbb{Z}\} = \{\ldots, \frac{1}{2^2}, \frac{1}{2}, 1, 2, 4, \ldots\}$.

H is not closed under addition since, for example, $1+2=3 \notin H$ but $1,2 \in H$.

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⑤ No. $\mathbb{Z}^*$ is not closed under inversion. For example $2 \in \mathbb{Z}^*$, but $\frac{1}{2} \notin \mathbb{Z}^*$.

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⑥ $H = \{2^n \mid n \in \mathbb{Z}\} \leq \mathbb{Q}^*$

proof:

closure: Let $x, y \in H$. Then $x = 2^k$ and $y = 2^l$ where $k, l \in \mathbb{Z}$. Then $xy = 2^{k+l} \in H$,

identity: $1 = 2^0 \in H$.

inverses: Let $x \in H$. Then $x = 2^k$ where $k \in \mathbb{Z}$. Then $\frac{1}{x} = 2^{-k} \in H$ and

$$x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x.$$

So, the inverse of x is in H.

(7) In $\mathbb{Z}$ we have

$$\langle 3 \rangle = \{\ldots, -3-3-3, -3-3, -3, 0, 3, 3+3, 3+3+3, \ldots\}$$
$$= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$$
$$= \{3n \mid n \in \mathbb{Z}\}$$

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(8) In $\mathbb{R}^*$ we have

$$\langle 5 \rangle = \{\ldots, \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}, \tfrac{1}{5}\cdot\tfrac{1}{5}, \tfrac{1}{5}, 1, 5, 5^2, 5^3, \ldots\}$$
$$= \{5^n \mid n \in \mathbb{Z}\}$$

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(9) In U_6 $\langle e^{2\pi i/3} \rangle = \{1, e^{2\pi i/3}, (e^{2\pi i/3})^2\}$

$$= \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$$

[since $(e^{2\pi i/3})^3 = 1$]

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(10) In U_8,

$$\langle e^{3\pi i/4} \rangle = \{1, e^{3\pi i/4}, e^{3\pi i/2}, e^{\pi i/4}, e^{\pi i}, e^{7\pi i/4}, e^{\pi i/2}, e^{5\pi i/4}\} = U_8$$

(11) In $\mathbb{Z}_8$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$

$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$

$\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \bar{2}, \bar{7}, \bar{4}, \bar{1}, \bar{6}, \bar{3}\} = \mathbb{Z}_8$

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(12) In D_{2n}, since the order of r is n we have

$\langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\}$

Let e be the identity of G.

(13) <u>closure:</u> Let $x, y \in HK$.

Then $x = h_1 k_1$ and $y = h_2 k_2$ where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then

$$xy = h_1 k_1 h_2 k_2 \underset{\text{since } G \text{ is abelian}}{=} (h_1 h_2)(k_1 k_2) \in HK$$

because $h_1 h_2 \in H$ since H is closed and $k_1 k_2 \in K$ since K is closed.

<u>identity:</u> Since $H \leq G$ we have that $e \in H$. Since $K \leq G$ we have that $e \in K$.

Hence, $e = e \cdot e \in HK$.

<u>inverses:</u> Let $x \in HK$. Then $x = hk$ where $h \in H$ and $k \in K$. Since $H \leq G$ we have that $h^{-1} \in H$. Since $K \leq G$ we have that $k^{-1} \in K$. Thus, $h^{-1} k^{-1} \in HK$.

Note that $(hk)^{-1} = k^{-1} h^{-1} \underset{\text{since } G \text{ is abelian}}{=} h^{-1} k^{-1}$.

So, $(hk)^{-1} \in HK$.

G is abelian

14) $H = \{ x \mid x^2 = e \}$

Then, $H \leq G$.

proof:

closure: Let $a, b \in H$. Then $a^2 = e$ and $b^2 = e$. Hence $(ab)^2 = abab \underset{\boxed{\text{since } G \text{ is abelian}}}{=} a^2 b^2 = e \cdot e = e$.

So, $ab \in H$.

identity: $e^2 = e$. Hence $e \in H$.

inverses: Let $a \in H$. Then $a^2 = e$.
So, $a^{-1}a^{-1}a^2 = a^{-1}a^{-1}e$. So, $e = a^{-1}a^{-1}$.
Thus, $e = (a^{-1})^2$. Thus, $a^{-1} \in H$.

15 closure: Let $a, b \in H$. Then $a = x^2$ and $b = y^2$ where $x, y \in G$. Thus,

$$ab = x^2 y^2 = xxyy = xyxy = (xy)^2.$$

since G is abelian

Since $xy \in G$ we have that $ab \in H$.

identity: $e = e^2$. Thus, $e \in H$.

inverses: Let $a \in H$. Then $a = x^2$ where $x \in G$. Then, $a^{-1} = (x^2)^{-1} = (xx)^{-1} = x^{-1}x^{-1} = (x^{-1})^2$. Hence, $a^{-1} \in H$, since $x^{-1} \in G$.

(16) <u>closure:</u> Let $a, b \in Z(G)$. Then
$ax = xa$ and $bx = xb$ for all $x \in G$.
Hence,
$$(ab)x = abx = axb = xab = x(ab)$$
for all $x \in G$.
Thus, $ab \in Z(G)$.

<u>identity:</u> Let $x \in G$. Then $ex = xe$.
Hence $e \in Z(G)$.

<u>inverses:</u> Let $a \in Z(G)$. ~~Then~~ Let $x \in G$.
Then $ax = xa$. Thus, $a^{-1}(ax)a^{-1} = a^{-1}(xa)a^{-1}$.
So, $xa^{-1} = a^{-1}x$. Hence, $a^{-1} \in Z(G)$.

(17) <u>closure:</u> Let $a, b \in H \cap K$.
Thus, $a, b \in H$ and $a, b \in K$.
Since $a, b \in H$ and H is a subgroup, we have that $ab \in H$.
Since $a, b \in K$ and K is a subgroup, we have that $ab \in K$.
Hence, $ab \in H \cap K$.

<u>identity:</u> $e \in H$ since $H \leq G$. Also, $e \in K$ since $K \leq G$. Thus, $e \in H \cap K$.

<u>inverses:</u> Let $a \in H \cap K$. Thus $a \in H$ and $a \in K$. Since $a \in H$ and $H \leq G$ we have that $a^{-1} \in H$. Since $a \in K$ and $K \leq G$ we have that $a^{-1} \in K$. Thus, $a^{-1} \in H \cap K$.