Homework #3 Solutions

(a) Note that plugging $x=-1$ into $2x+5$ gives $2(-1)+5=3$. Let's show that $\lim_{x\to -1}(2x+5)=3$.

① Let $\varepsilon>0$ be fixed.

Our goal is to find $\delta>0$ so that if

$\underbrace{0<|x-(-1)|<\delta}_{0<|x+1|<\delta}$ then $|(2x+5)-3|<\varepsilon$.

Note that (no matter what δ is)

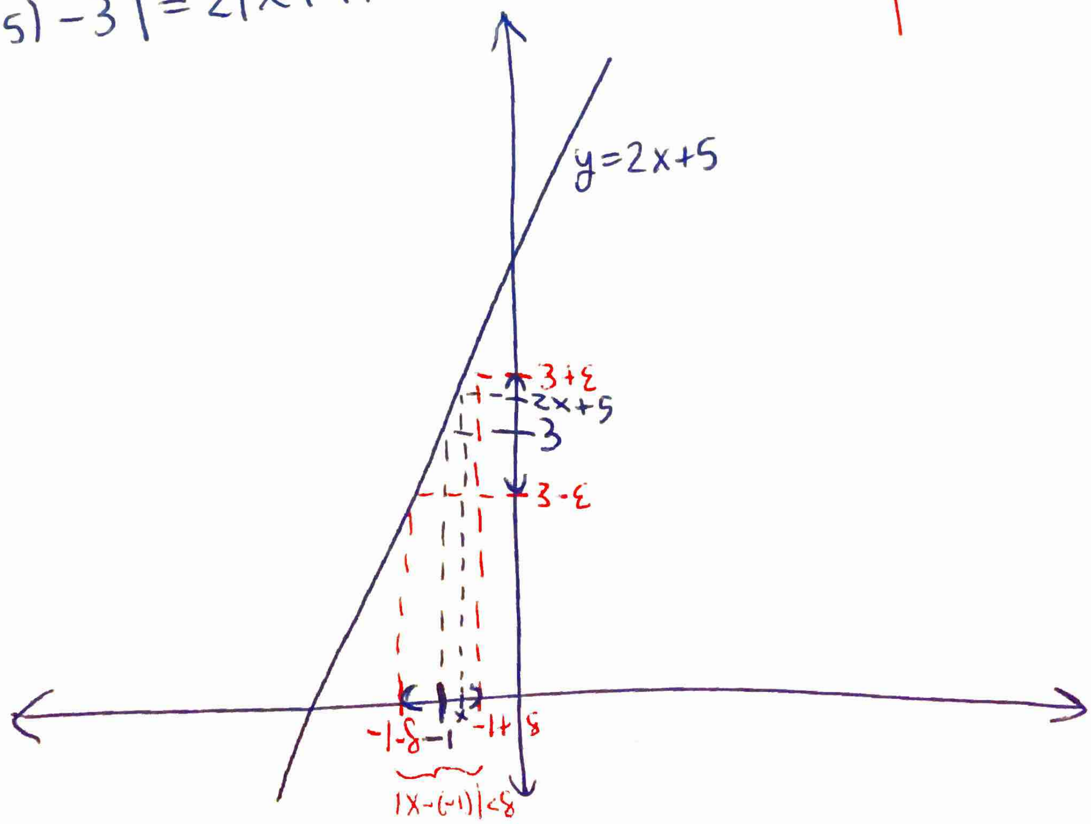
$|(2x+5)-3|=|2x+2|=|2||x+1|=\underbrace{2|x+1|}$.

We want this to be $<\varepsilon$. ie, $2|x+1|<\varepsilon$. so lets just make $|x+1|<\frac{\varepsilon}{2}$ happen so we set $\delta=$

Let $\delta=\frac{\varepsilon}{2}$.

Then if $0<|x+1|<\frac{\varepsilon}{2}$ ($\uparrow \delta$) we have that

$|(2x+5)-3|=2|x+1|<2\cdot\frac{\varepsilon}{2}=\varepsilon.$ $\blacksquare$



① (b) At $x = 1$, $\frac{5x}{x+3} = \frac{5}{4}$. Let's show that $\lim_{x \to 1} \frac{5x}{x+3} = \frac{5}{4}$

Let $\varepsilon > 0$ be fixed.

We want to find $\delta > 0$ so that if $\underbrace{0 < |x-1| < \delta}_{x \text{ is within } \delta \text{ distance of } 1}$

then $\underbrace{\left| \frac{5x}{x+3} - \frac{5}{4} \right| < \varepsilon}_{\frac{5x}{x+3} \text{ is within } \varepsilon \text{ distance of } 5/4}$.

oh yeah we've got $|x-1|$

Note that $\underbrace{\left| \frac{5x}{x+3} - \frac{5}{4} \right|}_{\text{Goal: Make } |x-1| \text{ appear in here somehow}} = \left| \frac{20x - 5x - 15}{4(x+3)} \right| = \left| \frac{15x - 15}{4x+12} \right| = \frac{15|x-1|}{|4x+12|}$

We first attack this part and get a bound on it by bounding $|x-1|$ by any number we choose.

Step 1:

Suppose $|x-1| < 1$ — I randomly picked this number

Then, $-1 < x-1 < 1$.

So, $0 < x < 2$. ← Now that we have x isolated, let's make $4x+3$ appear.

Thus, $0 < 4x < 8$

So, $12 < 4x+12 < 20$

Thus, $\frac{1}{12} > \frac{1}{4x+12} > \frac{1}{20}$.

$0 < \frac{1}{4x+3} < \frac{1}{3}$

So, $\left| \frac{1}{4x+12} \right| = \frac{1}{4x+12} < \frac{1}{12}$. And thus, $\frac{15|x-1|}{|4x+12|} < \frac{15}{12}|x-1|$.

$0 < 1/4x+3$

Want $\frac{15}{12}|x-1| < \varepsilon$ OR $|x-1| < \frac{\varepsilon}{15/12}$

Step 2:

Now, let $\delta = \min\left\{ \frac{\varepsilon}{(15/12)}, 1 \right\}$ — put this in here to keep the bound from step 1

$|x-1| < \delta \leq \varepsilon/(15/12)$

If $0 < |x-1| < \delta$, then

$$\left| \frac{5x}{x+3} - \frac{5}{4} \right| = \frac{15|x-1|}{|4x+12|} < \frac{15}{12}|x-1| < \frac{15}{12}\left(\frac{\varepsilon}{(15/12)} \right) = \varepsilon. \quad \blacksquare$$

Step 1: $|x-1| < \delta = \min\left\{ \frac{\varepsilon}{(15/12)}, 1 \right\} < 1$

1 (c) Note that when $x = 2$, we have that $x^4 = 2^4 = 16$.

Let's show $\lim_{x \to 2} x^4 = 16$.

Let $\varepsilon > 0$.

We want to find $\delta > 0$ so that if $\underbrace{0 < |x-2| < \delta}_{x \text{ is close to } 2}$

then $\underbrace{|x^4 - 16|}_{x^4 \text{ is close to } 16} < \varepsilon$.

Note that

$$\underbrace{|x^4 - 16|}_{\text{my goal right now is to make } |x-2| \text{ appear in this term somehow. Like this}} = |(x^2-4)(x^2+4)| = |(x-2)(x+2)(x^2+4)|$$

$$= |x-2| \underbrace{|x+2||x^2+4|}_{\text{now we bound these by bounding } |x-2|}$$

Step 1:

Suppose $|x-2| < 1$.

Note that $|x+2| = |x - 2 + 2 + 2| = |(x-2) + 4|$

$\leq |x-2| + |4|$

$< 1 + 4 = 5$.

Lets now bound $|x^2+4|$.

Since $|x-2| < 1$ we know $-1 < x-2 < 1$.

So, $1 < x < 3$. So, $1 < x^2 < 9$. Thus, $5 < x^2 + 4 < 13$.

Thus, $|x^2+4| < 13$.

THEREFORE, if $|x-2| < 1$, then

$$|x^4 - 16| = |x-2||x+2||x^2+4| < |x-2| \cdot 5 \cdot 13 = \underbrace{65|x-2|}_{\text{Now make this} < \varepsilon}.$$

Step 2: Let $\delta = \min\left\{1, \frac{\varepsilon}{65}\right\}$.

If $0 < |x-2| < \delta$, then

$$|x^4 - 16| \underset{\uparrow \text{ step 1}}{<} 65|x-2| \underset{\uparrow\ \delta \leq \frac{\varepsilon}{45}}{<} 65 \cdot \frac{\varepsilon}{65} = \varepsilon. \quad \blacksquare$$

① (d) From Calculus we know that

$$\lim_{x \to \infty} \frac{2x}{x^2+1} = \lim_{x \to \infty} \frac{\frac{2}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0.$$

Let's prove it!

Let $\varepsilon > 0$.

We need to find $N > 0$ where if $x \geq N$ then $\left| \frac{2x}{x^2+1} - 0 \right| < \varepsilon$.

Note that if $x > 0$ then

$$\left|\frac{2x}{x^2+1} - 0\right| = \left| \frac{2x}{x^2+1} \right| = \frac{2x}{x^2+1} < \frac{2x}{x^2} = \frac{2}{x}$$

Since $x \to \infty$ we may assume $x > 0$

And $\frac{2}{x} < \varepsilon$ iff $\frac{2}{\varepsilon} < x$.

Let $N = \frac{2}{\varepsilon}$.

If $x \geq N$, then $\left| \frac{2x}{x^2+1} - 0 \right| < \frac{2}{x} < \varepsilon$. ▨

① (e) When we plug $x=1$ into $\frac{1}{x^2}$ we get $\frac{1}{1^2}=1$.

Lets show that $\lim\limits_{x\to 1}\frac{1}{x^2}=1$.

Let $\varepsilon>0$.

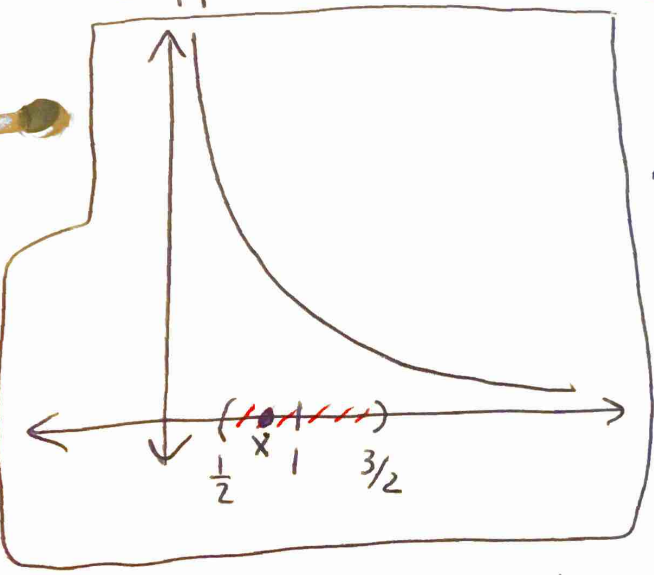
We need to find $\delta>0$ so that if $0<|x-1|<\delta$, then $\left|\frac{1}{x^2}-1\right|<\varepsilon$.

$\boxed{|1-x|=|-(1-x)|=|x-1|}$

Consider the equation

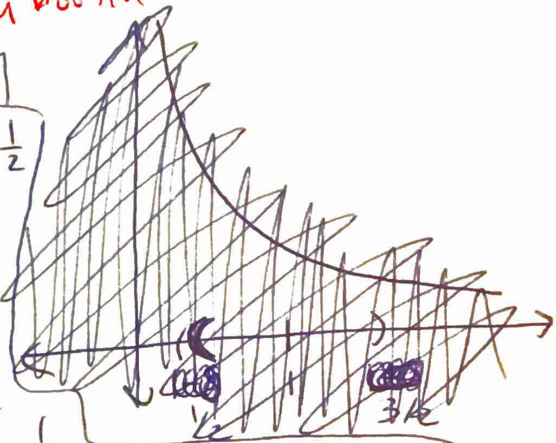
$$\left|\frac{1}{x^2}-1\right| = \left|\frac{1-x^2}{x^2}\right| = \frac{|1-x||1+x|}{|x^2|} = \frac{|x-1||1+x|}{|x^2|}.$$

Suppose that $|x-1|<\frac{1}{2}$ ← starter bound on δ.



Intuition:

If $|x-1|<\frac{1}{2}$ then we have this picture of where x lives. Note that we picked $\frac{1}{2}$ so x stays away from the y-axis where $\frac{1}{x^2}$ becomes infinite



Then $-\frac{1}{2}<x-1<\frac{1}{2}$.

So, $\frac{1}{2}<x<\frac{3}{2}$.

Thus, $|1+x|\le|1|+|x|<1+\frac{3}{2}=\frac{5}{2}$.

Also we have $\frac{1}{4}<x^2<\frac{9}{4}$.

Thus, $4>\frac{1}{x^2}>\frac{4}{9}$. So, $\left|\frac{1}{x^2}\right|<4$.

Therefore, if $|x-1| < \frac{1}{2}$, then

$$\left|\frac{1}{x^2}-1\right| = \frac{|x-1|\,|1+x|}{|x^2|} < \left(|x-1| \cdot \frac{5}{2}\right) \cdot 4$$

$|1+x| < \frac{5}{2}$

$\frac{1}{|x^2|} < 4$

$$= 10 \cdot |x-1|.$$

Let $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{10}\right\}$.

If $0 < |x-1| < \delta$, then

$$\left|\frac{1}{x^2}-1\right| < 10|x-1| < 10 \cdot \frac{\varepsilon}{10} = \varepsilon \qquad \blacksquare$$

since $|x-1| < \frac{1}{2}$

since $|x-1| < \frac{\varepsilon}{10}$

So, if $|x-1|<1$, then

$$\left|\frac{1}{x^2}-1\right| = \frac{|x-1||1+x|}{|x^2|} < \frac{|x-1|\cdot 3}{4} = \underbrace{\frac{3}{4}|x-1|}$$

now we bound this to be $<\varepsilon$

$\frac{3}{4}|x-1|<\varepsilon$

if $|x-1| < \varepsilon/(3/4) = \frac{4\varepsilon}{3}$.

Take $\delta = \min\left\{1, \frac{4\varepsilon}{3}\right\}$

first bound on $|x-1|$

further bound if needed for this part if 1 isn't small enough to do the trick

Then, if $0<|x-1|<\delta$, we have

$$\left|\frac{1}{x^2}-1\right| < \frac{3}{4}|x-1| < \frac{3}{4}\cdot\frac{4\varepsilon}{3} = \varepsilon. \quad \blacksquare$$

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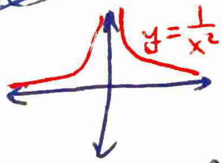
① (f). Looking at the graph of $y=\frac{1}{x^2}$

we have 

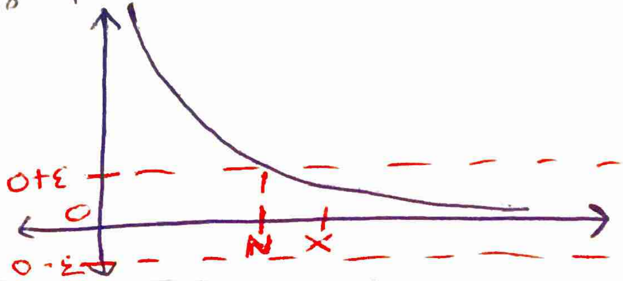
So, we think that $\lim_{x\to\infty}\frac{1}{x^2}=0$.

Lets prove it using the definition of limit.

Let $\varepsilon>0$ be fixed.

We need to find N so that if $x>N$

then $\left|\underbrace{\frac{1}{x^2}}_{\text{function}} - \underbrace{0}_{\text{limit}}\right| < \varepsilon$.



Note that $\left|\frac{1}{x^2}\right| < \varepsilon$ iff $\frac{1}{x^2} < \varepsilon$ iff $\frac{1}{\varepsilon} < x^2$ $\Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$.

Assume $x > 0$ since we are letting x go to ∞.

Let $N > \frac{1}{\sqrt{\varepsilon}}$. If $x \geq N$, then by the above argument we have that $\left|\frac{1}{x^2} - 0\right| < \varepsilon$. ▨

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① (g) Plugging $x = 2$ into $x^3 - 1$ we get 7.

We will prove that $\lim_{x \to 2} (x^3 - 1) = 7$.

Let $\varepsilon > 0$.

We want to find $\delta > 0$ so that if $\underbrace{0 < |x-2| < \delta}_{x \text{ is close to } 2}$

then $\underbrace{|(x^3-1) - 7| < \varepsilon}_{x^3-1 \text{ is close to } 7}$.

Yes! We've done it. There's our $|x-2|$

Note that

$|(x^3-1) - 7| = |x^3 - 8| = |(x-2)(x^2+2x+4)| = |x-2|\underbrace{|x^2+2x+4|}$

Goal: Make $|x-2|$ appear in here so we can use the δ bound

Divide out $x-2$ if we can:

$$\begin{array}{r}
x^2+2x+4 \\
x-2 \overline{)\, x^3 - 8} \\
-(x^3 - 2x^2) \\
\hline
2x^2 - 8 \\
-(2x^2 - 4x) \\
\hline
4x - 8 \\
-(4x - 8) \\
\hline
0
\end{array}$$

We now bound this part

Suppose $|x-2|<1$ ← A random bound I picked so we can start bounding $|x-2||x^2+2x+4|$

Then, $-1<x-2<1$.

So, $1<x<3$. ← We now have a bound on x

Then, $|x^2+2x+4| \le |x^2|+|2x|+|4|$

$= |x^2|+2|x|+4$

$< 3^2+2\cdot 3+4 = 19$

$x<3$

So, if $|x-2|<1$, then $|x-2||x^2+2x+4| < 19|x-2|$

Want $19|x-2|<\varepsilon$ or $|x-2|<\frac{\varepsilon}{19}$

Let $\delta = \min\{1, \frac{\varepsilon}{19}\}$.

initial bound

bound from here

If $|x-2|<\delta$, then

$|(x^3-1)-7| = |x-2||x^2+2x+4| < 19|x-2| < 19\cdot\frac{\varepsilon}{19} = \varepsilon$.

$\blacksquare$

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① h) We show that $\lim_{x\to c}(ax+b) = ac+b$, where $a\neq 0$.

Let $\varepsilon>0$.

Note that $|(ax+b)-(ac+b)| = |ax-ac| = |a||x-c|$. Let $\delta = \frac{\varepsilon}{|a|}$.

Suppose that $|x-c|<\delta$, then

$|(ax+b)-(ac+b)| = |a||x-c| < |a|\cdot\frac{\varepsilon}{|a|} = \varepsilon$.

$\blacksquare$

1 (i) We show $\lim_{x \to \infty} \frac{1}{x^a} = 0$.

Let $\varepsilon > 0$.

Note that

$$\left| \frac{1}{x^a} - 0 \right| = \left| \frac{1}{x^a} \right| = \frac{1}{x^a}$$

We can assume $x > 0$ since x is going to ∞

And $\frac{1}{x^a} < \varepsilon$ iff $\frac{1}{\varepsilon} < x^a$ iff $\left(\frac{1}{\varepsilon}\right)^{1/a} < x$.

Let $N > \left(\frac{1}{\varepsilon}\right)^{1/a}$.

If $x \geq N$, then

$$\left| \frac{1}{x^a} - 0 \right| = \frac{1}{x^a} < \varepsilon.$$

by above

$\blacksquare$

② (a) Let $\varepsilon = 1$.

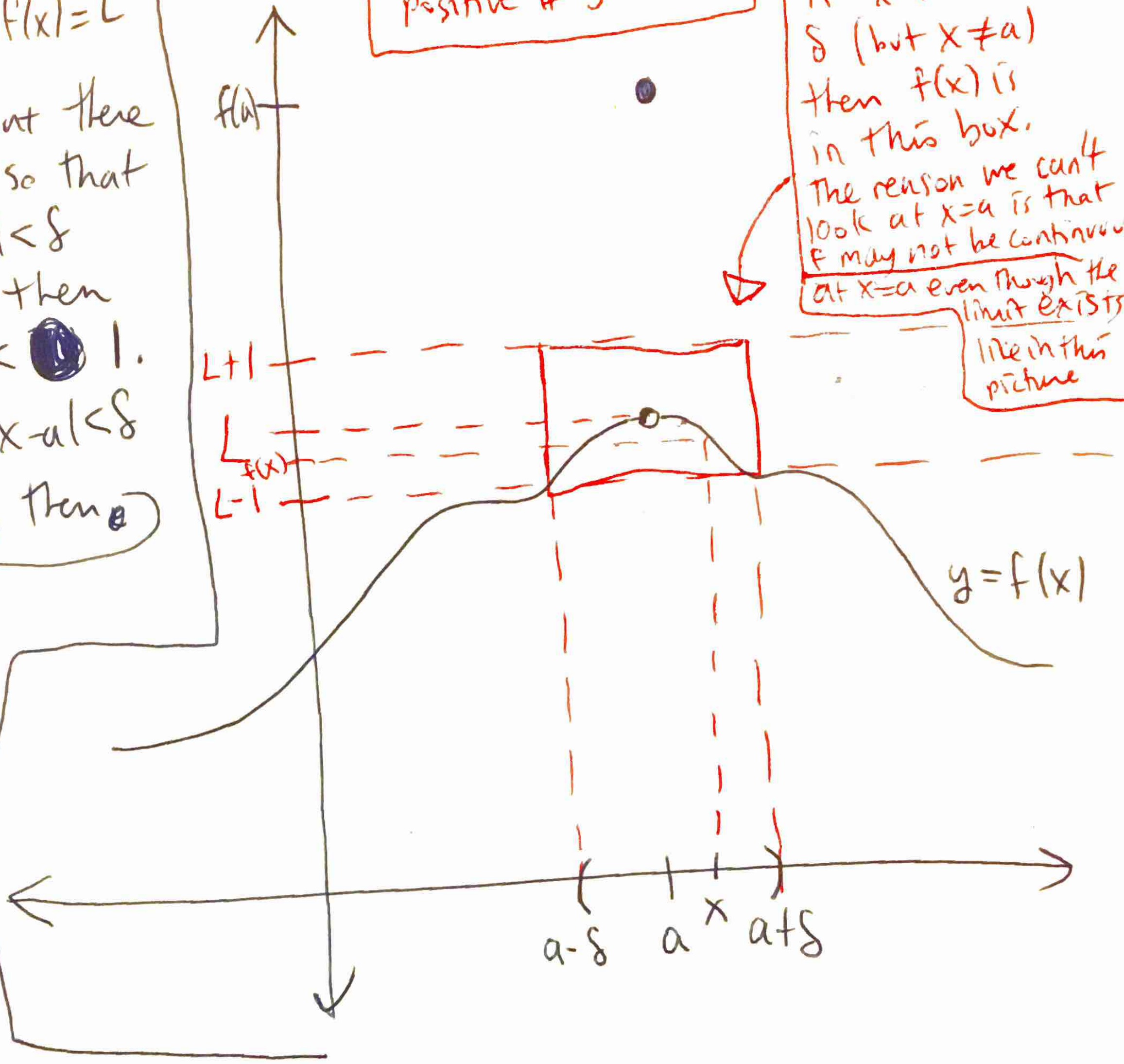
Since $\lim_{x \to a} f(x) = L$ we know that there exists $\delta > 0$ so that if $0 < |x-a| < \delta$ and $x \in D$ then $|f(x) - L| < 1$.

So, if $0 < |x-a| < \delta$ and $x \in D$, then

I chose $\varepsilon = 1$ arbitrarily. You can pick any positive # you want

What we are saying is that if x is close to δ (but $x \neq a$) then $f(x)$ is in this box.

The reason we can't look at $x=a$ is that f may not be continuous at $x=a$ even though the limit exists like in this picture



$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L|$$
$$< 1 + |L|$$

Take $M = 1 + |L|$.

So, if $0 < |x-a| < \delta$ and $x \in D$, then $|f(x)| < M$. ▨

$\boxed{2(b)}$ We show that $\dfrac{1}{(x-2)^2}$ is unbounded near 2.

Let $M>0$ be a fixed real number.

We want to show that $\left|\dfrac{1}{(x-2)^2}\right| > M$ for some x That is close to 2 (but $x \neq 2$).

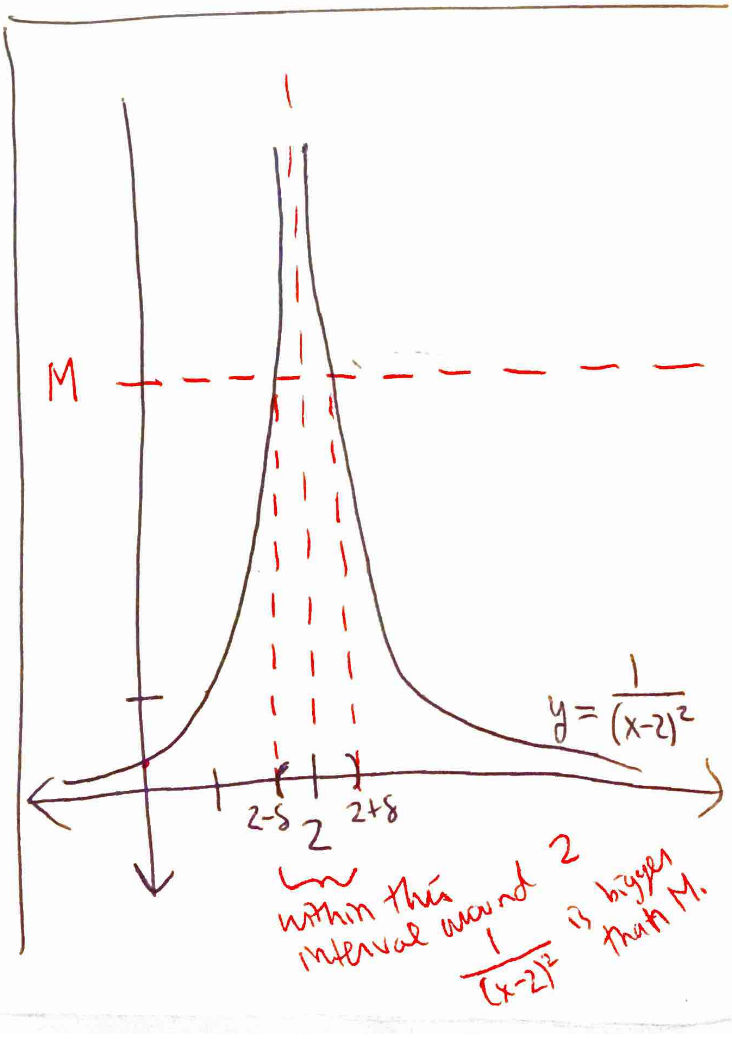
Note that $\left|\dfrac{1}{(x-2)^2}\right| > M$ iff $\dfrac{1}{M} > |(x-2)^2|$

iff $\dfrac{1}{M} > |x-2|^2$ iff $\dfrac{1}{\sqrt{M}} > |x-2|$.

Let $\delta = \dfrac{1}{\sqrt{M}}$.

If $0<|x-2|<\delta$, then by the above calculations $\left|\dfrac{1}{(x-2)^2}\right| > M$.

Hence, no matter how big we make M, $\dfrac{1}{(x-2)^2} > M$ in a neighborhood around 2.



③ (a)

Let $\varepsilon = 1$.

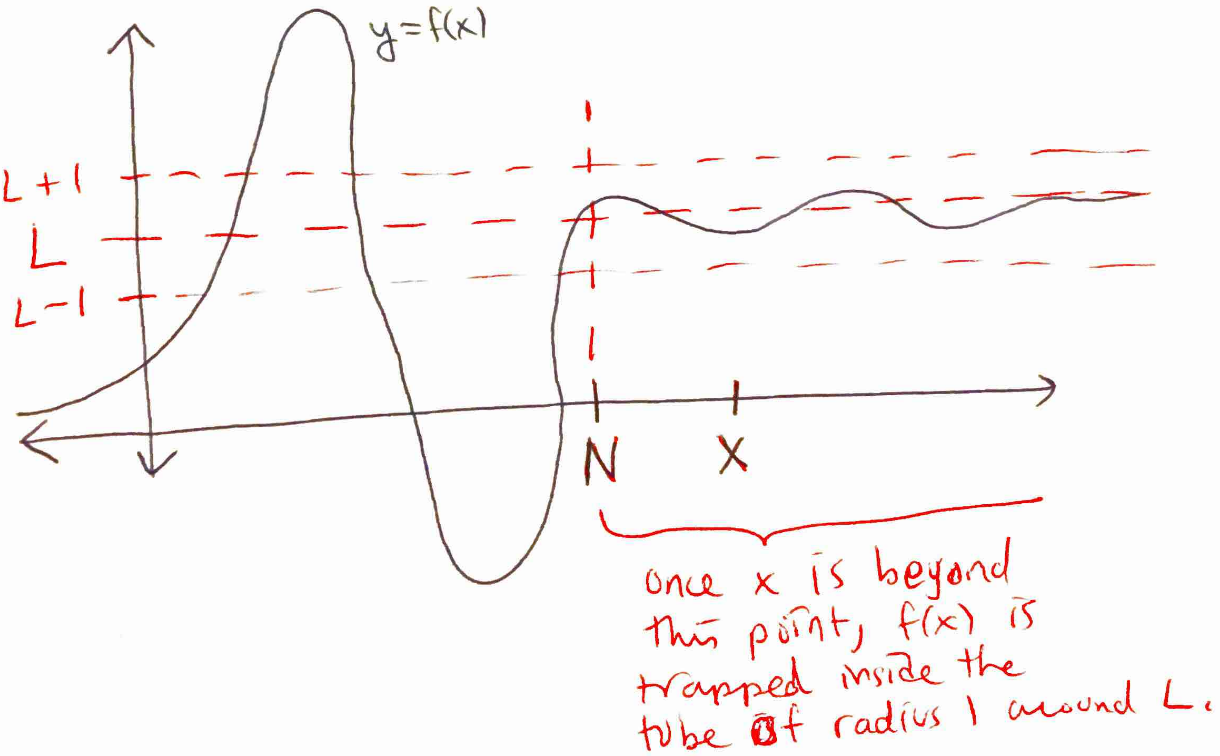
I randomly picked $\varepsilon = 1$. You can pick any positive # you want.

Since $\lim_{x\to\infty} f(x) = L$, we know that there exists $N > 0$ so that if $x \geq N$, then $|f(x) - L| < 1$.

So, if $x \geq N$, then

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + |L|.$$

Take $C = 1 + |L|$. $\blacksquare$



③ (b) We show that x^3-1 is unbounded as x goes to ∞. That is, we show that x^3-1 does not satisfy 3(a).

Let $C>0$ be any real number.

Let's find some point forward and where $|x^3-1|>C$.

Assume $x^3-1>0$ (since $x\to\infty$ we can do this)

Then,

$|x^3-1|>C$ iff $x^3-1>C$ iff $x^3>C+1$ iff $x>(C+1)^{1/3}$.

Therefore, if $x>(C+1)^{1/3}$, then $x^3-1>C$.

So, x^3-1 is unbounded as x goes to ∞.

This contradicts 3(a). So $\lim_{x\to\infty} x^3-1$ does not exist.

(4)(a) Let $\varepsilon > 0$.

Since $\lim_{x\to\infty} f(x) = L$, there exists $N > 0$ so that if $x \geq N$, then $|f(x) - L| < \varepsilon$.

Since a_n is unbounded and increasing, there exists $M > 0$ so that $a_n \geq N$ for all $n \geq M$.

Thus, if $n \geq M$, then $|f(a_n) - L| < \varepsilon$.

Therefore, $(f(a_n))$ converges to L. ∎

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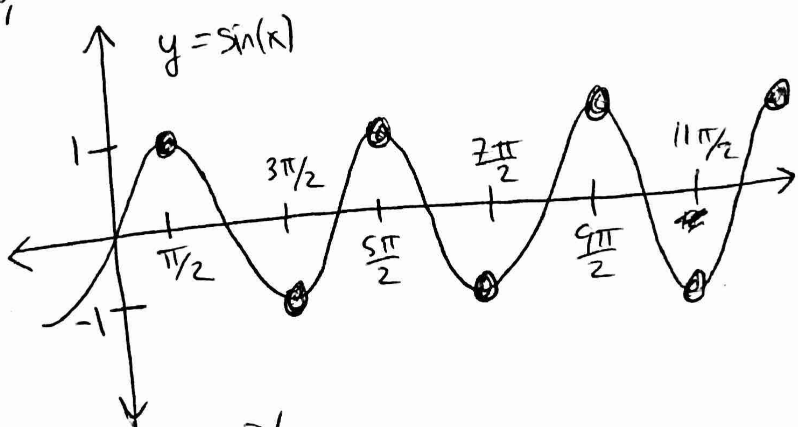
(4)(b) We show that $\lim_{x\to\infty} \sin(x)$ does not exist.

Let (a_n) be the sequence

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \ldots$$

Then a_n is increasing and unbounded.

And $\sin(a_n) = (-1)^{n+1}$ which does not converge to any limit as $n \to \infty$.



Thus, $\sin(x)$ does not satisfy 4(a).

Hence $\lim_{x\to\infty} \sin(x)$ does not exist.

5 Let $\varepsilon > 0$ be fixed.

Note that

$$\begin{aligned} |f(x)g(x) - AB| &= |f(x)g(x) - f(x)\cdot B + f(x)\cdot B - AB| \\ &\le |f(x)g(x) - f(x)\cdot B| + |f(x)\cdot B - AB| \\ &= |f(x)|\,|g(x) - B| + |B|\,|f(x) - A|. \end{aligned}$$

By exercise 2(a) there exists $M > 0$ and $\delta_1 > 0$ so that if $x \in D$ and $0 < |x-a| < \delta_1$, then $|f(x)| < M$.

Let $C = \max\{|B|, 1\}$.

This is in here in case $B=0$ because we are going to divide by C below.

Since $\lim_{x\to a} g(x) = B$, there exists $\delta_2 > 0$ so that if $x \in D$ and $0 < |x-a| < \delta_2$, then $|g(x) - B| < \frac{\varepsilon}{2M}$.

Since $\lim_{x\to a} f(x) = A$, there exists $\delta_3 > 0$ so that if $x \in D$ and $0 < |x-a| < \delta_3$, then $|f(x) - A| < \frac{\varepsilon}{2C}$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $x \in D$ and $0 < |x-a| < \delta$ then (all the above facts will be true since δ is smaller than all the δ_i or equal to)

$$\begin{aligned} |f(x)g(x) - AB| &\le |f(x)|\,|g(x) - B| + |B|\,|f(x) - A| \\ &< M\cdot|g(x) - B| + C\cdot|f(x) - A| \\ &< M\cdot\frac{\varepsilon}{2M} + C\cdot\frac{\varepsilon}{2C} = \varepsilon. \end{aligned}$$

$\blacksquare$