Math 465 - Homework # 3
Limits of functions

1. Prove the following limit exists using the ϵ - definition of limit.
   (a) $\lim_{x \to -1} 2x + 5$
   (b) $\lim_{x \to 1} \dfrac{5x}{x+3}$
   (c) $\lim_{x \to 2} x^4$
   (d) $\lim_{x \to \infty} \dfrac{2x}{x^2+1}$
   (e) $\lim_{x \to 1} \dfrac{1}{x^2}$
   (f) $\lim_{x \to \infty} \dfrac{1}{x^2}$
   (g) $\lim_{x \to 2} x^3 - 1$
   (h) $\lim_{x \to c} ax + b$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$.
   (i) $\lim_{x \to \infty} \dfrac{1}{x^a}$ where a is a fixed real number with $a > 0$.

2. (a) Let $f : D \to \mathbb{R}$ be a function. Suppose that $\lim_{x \to a} f(x)$ exists for some $a \in \mathbb{R}$. Show that f is bounded near a (but not necessarily at $x = a$). That is, show that there exists $M > 0$ and $\delta > 0$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $|f(x)| \leq M$.
   (b) Show that $\lim_{x \to 2} \dfrac{1}{(x-2)^2}$ does not exist.

3. (a) Let $f : [a, \infty) \to \mathbb{R}$ be a function for some $a \in \mathbb{R}$. Suppose that $\lim_{x \to \infty} f(x)$ exists. Prove: There exists an $C > 0$ and an $N > 0$ such that $|f(x)| < C$ for all $x \geq N$.
   (b) Show that $\lim_{x \to \infty} x^3 - 1$ does not exist.

4. (a) Suppose that $\lim_{x \to \infty} f(x)$ exists and is equal to a real number L. Show that if (a_n) is any unbounded increasing sequence of real numbers, then the sequence $(f(a_n))$ converges to L.

(b) Show that $\lim_{x\to\infty} \sin(x)$ does not exist.

5. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Let a a limit point of D. Suppose that $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$. Prove that $\lim_{x\to a} f(x)g(x) = AB$.