Homework 4 Solutions

(1)

(a) φ is not a ring homomorphism.
For example,

$$\varphi(2\cdot 3) = \varphi(6) = 2\cdot 6 = 12$$

but

$$\varphi(2)\cdot\varphi(3) = (2\cdot 2)(2\cdot 3) = 24.$$

Note: φ is a group homomorphism under $+$ because if $x, y \in \mathbb{Z}$ then

$$\varphi(x+y) = 2(x+y) = 2x+2y = \varphi(x)+\varphi(y).$$

(b) φ is not a ring homomorphism. φ satisfies $\varphi(AB) = \varphi(A)\varphi(B)$. You can check this. However, it does not satisfy the additive property $\varphi(A+B) = \varphi(A)+\varphi(B)$. For example,

$$\varphi\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}\right) = \varphi\begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$$

but

$$\varphi\begin{pmatrix}1&0\\0&0\end{pmatrix}+\varphi\begin{pmatrix}0&0\\0&1\end{pmatrix} = 0+0 = 0$$

φ is NOT a ring homomorphism.

(c) Let $a,b,c,d,e,f,g,h \in \mathbb{R}$, Then φ satisfies

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \varphi\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$= (a+e)+(d+h)$$

$$= (a+d)+(e+h) = \varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \varphi\begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

However, φ does not satisfy the multiplicative property. For example,

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

but

$$\varphi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\varphi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot 1 = 1.$$

(d) φ is a ring homomorphism.

Let $a,b,c,d \in \mathbb{Z}$. Then

$$\varphi((a,b)+(c,d)) = \varphi(a+c, b+d) = a+c$$

$$= \varphi(a,b) + \varphi(c,d)$$

and

$$\varphi((a,b)(c,d)) = \varphi(ac, bd) = ac = \varphi(a,b)\varphi(c,d)$$

We have that

$$\ker(\varphi) = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid \varphi(a,b) = 0 \}$$

$$= \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = 0 \}$$

$$= \{ (0,b) \mid b \in \mathbb{Z} \}$$

$$= \{ \ldots, (0,-2), (0,-1), (0,0), (0,1), (0,2), \ldots \}$$

φ is not an isomorphism. It is onto, but not 1-1. For example,

$$\varphi(1,0) = 1 \quad \text{and} \quad \varphi(1,1) = 1$$

but $(1,0) \neq (1,1)$.

Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is a ring homomorphism. Let's show that φ cannot be an isomorphism. Note that $\varphi(1)^2 = \varphi(1)\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)$. So, $\varphi(1)[\varphi(1) - 1] = 0$. Since our equation is in $\mathbb{C}$, either $\varphi(1) = 0$ or $\varphi(1) = 1$.

If $\varphi(1) = 0$, then φ is not 1-1 since $\varphi(0) = 0$ also.

If $\varphi(1) = 1$, then $\varphi(-1) = -\varphi(1) = -1$. If φ were an isomorphism, then $\exists r \in \mathbb{R}$ such that $\varphi(r) = i$. Then $\varphi(r)^2 = i^2 = -1$. So, $\varphi(r^2) = -1$. But then, since φ is 1-1, $r^2 = -1$. But this can't happen since $r \in \mathbb{R}$.

So, in either case, φ is not an isomorphism.

~~(⇒) If $a^2-b^2=(a+b)(a-b)$, then~~

~~$a^2-b^2=(a+b)(a-b)=a^2-ab+ba-b^2$. So,~~

~~$-a^2+(a^2-b^2)+b^2 = -a^2+(a^2-ab+ba-b^2)+b^2$. Thus,~~

~~$0=-ab+ba$. Therefore, $ab=ba$.~~

~~(⇐) Go backwards in part (⇒).~~

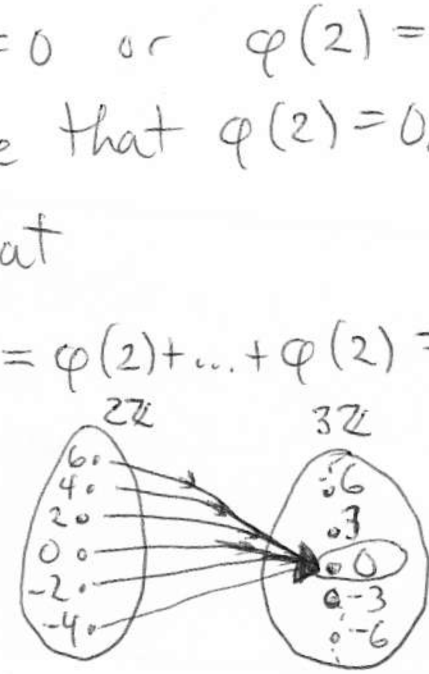
3

<u>$2\mathbb{Z} \not\cong 3\mathbb{Z}$ as rings:</u> Suppose $\varphi: 2\mathbb{Z} \to 3\mathbb{Z}$ is a ring homomorphism. Consider $\varphi(2)$. What is it? Note that $\varphi(2^2)=\varphi(2)\varphi(2)=\varphi(2)^2$ and also $\varphi(2^2)=\varphi(4)=\varphi(2+2)=\varphi(2)+\varphi(2)$. So, $\varphi(2)^2=\varphi(2)+\varphi(2)=2\varphi(2)$. So, $\varphi(2)\left[\varphi(2)-2\right]=0$.

Since $3\mathbb{Z}$ is an integral domain, our equation gives that either $\varphi(2)=0$ or $\varphi(2)=2$. Since $2\notin 3\mathbb{Z}$, we must have that $\varphi(2)=0$. Then if $2n\in 2\mathbb{Z}$, we have that

$$\varphi(2n)=\varphi(\underbrace{2+2+\ldots+2}_{n\text{ times}})=\varphi(2)+\ldots+\varphi(2)=0.$$

So our map looks like



That is, $\varphi(2n)=0$ $\forall n\in\mathbb{Z}$. So φ can't be an isomorphism.

(4) (a) We use the subring criteria. I will show that R_2 is a subring of $M_2(\mathbb{R})$. You show that R_1 is a subring of $\mathbb{R}$.

- $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\cdot 0 \\ 0 & 0 \end{pmatrix} \in R_2$ (set $a=0, b=0$)
- Let $a,b,c,d \in \mathbb{Z}$. Then

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} - \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} (a-c) & 2(b-d) \\ (b-d) & (a-c) \end{pmatrix} \in R_2$$

and

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+2bd & 2ad+2bc \\ bc+ad & 2bd+ac \end{pmatrix}$$

$$= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix} \in R_2$$

So, R_2 is a subring of $M_2(\mathbb{R})$.

(b) Let $\varphi : R_1 \to R_2$ be given by

$$\varphi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}.$$

Let $a,b,c,d \in \mathbb{Z}$. Then

$$\varphi\left((a+b\sqrt{2}) + (c+d\sqrt{2})\right) = \varphi\left((a+c)+(b+d)\sqrt{2}\right) = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$$

and $\varphi(a+b\sqrt{2}) + \varphi(c+d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$

Thus φ satisfies the additive property of a ring homomorphism.

Also,

$$\varphi((a+b\sqrt{2})(c+d\sqrt{2})) = \varphi((ac+2bd)+(ad+bc)\sqrt{2})$$

$$= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}$$

and

$$\varphi(a+b\sqrt{2})\varphi(c+d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$$

$$= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}$$

Thus, φ satisfies the multiplicative property of a ring homomorphism.

φ is 1-1:

Note that

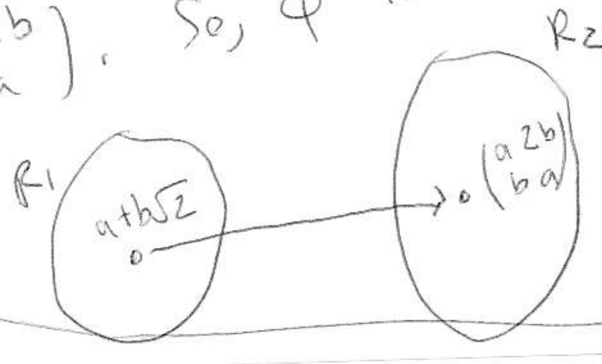
$$\ker(\varphi) = \left\{ a+b\sqrt{2} \;\middle|\; a,b \in \mathbb{Z} \text{ and } \varphi(a+b\sqrt{2}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ a+b\sqrt{2} \;\middle|\; a,b \in \mathbb{Z} \text{ and } \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \{0+0\sqrt{2}\} = \{0\}.$$

Hence φ is 1-1 by a Theorem from class.

φ is onto: Let $M \in R_2$. Then $M = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ for some $a,b \in \mathbb{Z}$. Then $a+b\sqrt{2} \in R_1$ and $\varphi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$. So, φ is onto.



Thus φ is an isomorphism. So, R_1 and R_2 are isomorphic.

~~Section 18 Solutions~~

~~15 $(1,1), (1,-1), (-1,1), (-1,-1)$~~

~~16 $\bar{1}, \bar{2}, \bar{3}, \bar{4}$~~

~~17 All nonzero elements of $\mathbb{Q}$~~

5 Suppose $\varphi : \mathbb{Z} \to \mathbb{Z}$ is a ring homomorphism.

What is $\varphi(1)$? Consider

$$\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)\varphi(1).$$

So, $\varphi(1)^2 - \varphi(1) = 0$. Thus, $\overbrace{\varphi(1)\left[\varphi(1) - 1\right] = 0}^{\text{(this is an equation in } \mathbb{Z}\text{)}}$.

Since $\mathbb{Z}$ is an integral domain, either $\varphi(1) = 0$ or $\varphi(1) - 1 = 0$. So,

either $\varphi(1) = 0$ or $\varphi(1) = 1$

What is $\varphi(n)$ equal to? If $n > 0$, then

$$\varphi(n) = \varphi(1+1+1+\dots+1) = \varphi(1) + \dots + \varphi(1) =$$

$$= \begin{cases} 0 & \text{if } \varphi(1) = 0 \\ n & \text{if } \varphi(1) = 1 \end{cases}$$

~~So, either $\varphi(n) = 0$ for all $n \in \mathbb{Z}$~~

~~$\varphi(n) = n$ for all $n \in \mathbb{Z}$.~~

~~Thus, φ is an isomorphism. So, R_1 and R_2 are isomorphic as rings.~~

(5) (continued...) By a thm in class, $\varphi(0)=0$.

If $n<0$, then by the previous page

$$\varphi(n)=\varphi(-n)=\begin{cases} 0 & \text{if } \varphi(1)=0 \\ -n & \text{if } \varphi(1)=1 \end{cases}$$

Hence, if $\varphi(1)=0$, then $\varphi(x)=0$ for all $x\in\mathbb{Z}$. If $\varphi(1)=1$, then $\varphi(x)=x$ for all $x\in\mathbb{Z}$. So, there are two ring homomorphisms $\varphi:\mathbb{Z}\to\mathbb{Z}$.

(6)

(a) Certainly ~~[illegible]~~

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$. (Just set $a=0$ and $b=0$.)

Let $a, b, c, d \in \mathbb{Z}$. Then

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a-c & 0 \\ 0 & b-d \end{pmatrix} \in R$$

and

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} \in R.$$

By the subring criteria, R is a subring of $M_2(\mathbb{R})$.

(b) Define the function $\varphi : R \to \mathbb{Z} \times \mathbb{Z}$ by $\varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (a, b)$.

<u>φ is a ring homomorphism</u>: Let $a, b, c, d \in \mathbb{Z}$.

Then $\varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = \varphi\begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} = (a+c, b+d)$

$$= (a,b) + (c,d) = \varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \varphi\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

and

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = \varphi\begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} = (ac, bd)$$

$$= (a,b)(c,d) = \varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \varphi\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

<u>φ is 1-1</u>: Let $a, b, c, d \in \mathbb{Z}$, suppose $\varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \varphi\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$.
Then $(a,b) = (c,d)$. Hence $a = c$ and $b = d$.
Thus, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$.

$\underline{\varphi \text{ is onto:}}$ Let $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. Then $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$ and $\varphi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (a,b)$. Thus φ is onto.

So, φ is an isomorphism. Thus, R and $\mathbb{Z} \times \mathbb{Z}$ are isomorphic as rings.

(7)

(a) We have that $\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$.

Thus,

$$\underbrace{-\varphi(0) + \varphi(0)}_{0'} = \underbrace{-\varphi(0) + \varphi(0)}_{0'} + \varphi(0)$$

So, $0' = \varphi(0)$.

(b) The statement $-\varphi(a) = \varphi(-a)$ means that $\varphi(-a)$ is the additive inverse for $\varphi(a)$. This can be shown by showing that

$$\varphi(a) + \varphi(-a) = \varphi(-a) + \varphi(a) = 0'.$$

Indeed, we have that

$$\varphi(a) + \varphi(-a) = \varphi(a + (-a)) = \varphi(0) = 0'$$

and

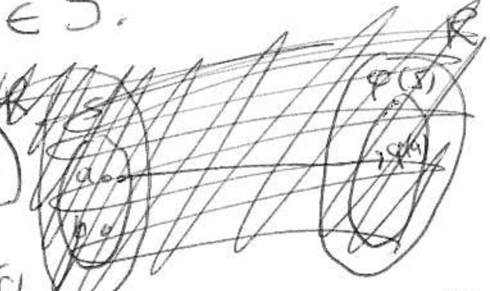
$$\varphi(-a) + \varphi(a) = \varphi((-a) + a) = \varphi(0) = 0'.$$

(c) We use the subring criteria.

Note that $0' = \varphi(0)$ and $0 \in S$.

Thus, $0' \in \varphi(S) = \{\varphi(x) \mid x \in S\}$.

Let $a, b \in \varphi(S)$. Then $a = \varphi(x)$ and $b = \varphi(y)$ where $x, y \in S$.



Note that $x-y \in S$ and $xy \in S$ since S is a subring of R.

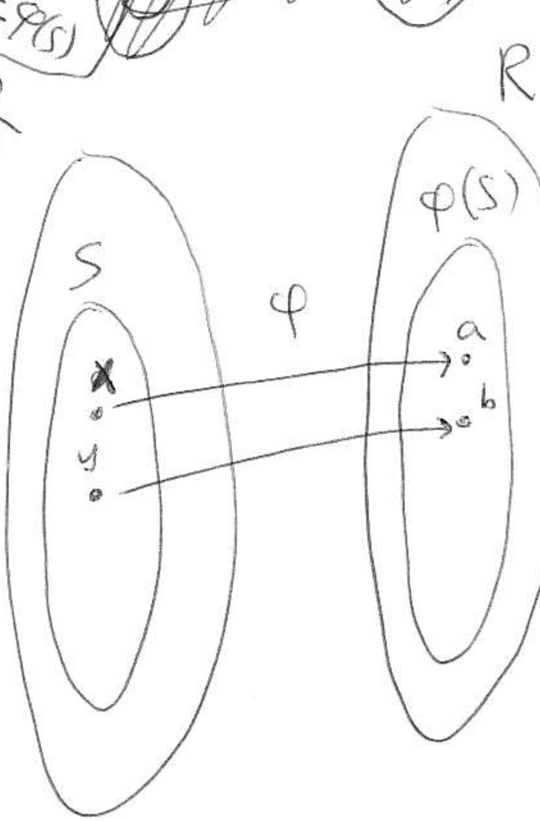
Then

$$a-b = \varphi(x) - \varphi(y) = \varphi(x-y) \in \varphi(S)$$

and

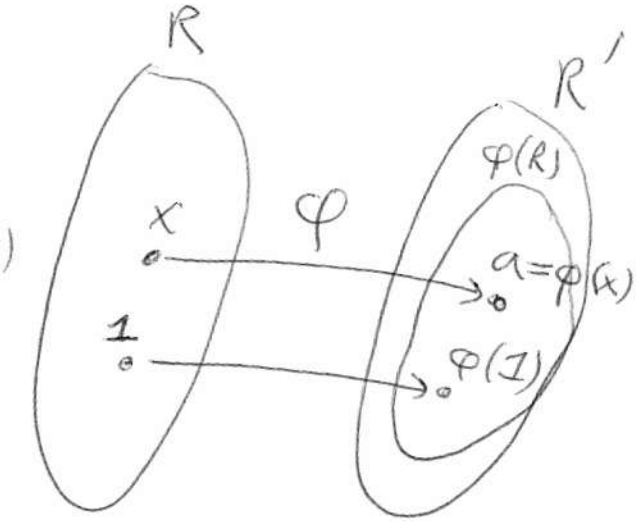
$$ab = \varphi(x)\varphi(y) = \varphi(xy) \in \varphi(S).$$

So, $\varphi(S)$ is a subring of R'.



(d) Let $a \in \varphi(R)$. Then $a = \varphi(x)$ where $x \in R$.

Then,



$$\varphi(1) \cdot a = \varphi(1) \cdot \varphi(x) = \varphi(1 \cdot x) = \varphi(x) = a$$

and

$$a \cdot \varphi(1) = \varphi(x) \cdot \varphi(1) = \varphi(x \cdot 1) = \varphi(x) = a.$$

Thus, $\varphi(1)$ is a multiplicative identity for $\varphi(R)$.