Homework #4 Solutions

① Let $\varepsilon > 0$ be fixed. Note that $2^2 - 1 = 3$.

Note that

$$|(x^2-1)-3| = |x^2-4| = \underbrace{|x-2||x+2|}_{\text{Goal: Make this } < \varepsilon \text{ by bounding } |x-2| \text{ appropriately}}.$$

Suppose $|x-2| < 1$.

arbitrary starting bound on $|x-2|$. Pick whatever positive # you like.

Then $-1 < x-2 < 1$.

So, $1 < x < 3$.

Thus, $|x+2| \le |x| + |2| < 3+2 = 5$.

So, if $|x-2|<1$, then

$$|(x^2-1)-3| = |x-2||x+2| < 5|x-2|.$$

This will be $< \varepsilon$ if $5|x-2| < \varepsilon$ or $|x-2| < \frac{\varepsilon}{5}$

Let $\delta = \min\{\frac{\varepsilon}{5}, 1\}$.

this 1 in here keeps the bounds above that we got on $|x+2|$

If $|x-2| < \delta$, then

$$|(x^2-1)-3| < 5|x-2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

So, if $|x-2| < \delta$, then $|(x^2-1)-3| < \varepsilon$. ▨

② Let $\varepsilon > 0$ be fixed. Note that $3(1)^2+1=4$.

Note that

$$|(3x^2+1)-4| = |3x^2-3| = |3||x^2-1|$$
$$= \underbrace{3|x-1||x+1|}_{\boxed{\text{Goal: Make This } < \varepsilon}}.$$

Suppose $|x-1| < 1$. arbitrary starting bound on $|x-1|$

Then, $-1 < x-1 < 1$.

So, $0 < x < 2$.

So, $1 < x+1 < 3$.

Thus, if $|x-1| < 1$, then $3|x-1||x+1| < 3|x-1|\cdot 3$
$= 9|x-1|$.

comes from

Want $9|x-1| < \varepsilon$ or $|x-1| < \frac{\varepsilon}{9}$

Let $\delta = \min\left\{\frac{\varepsilon}{9}, 1\right\}$.

get at least the starting bound

If $|x-1| < \delta$, then

$$|(3x^2+1)-4| = 3|x-1||x+1| < 9|x-1| < 9\cdot\frac{\varepsilon}{9} = \varepsilon.$$

∎

③ Let $\varepsilon > 0$ be fixed.

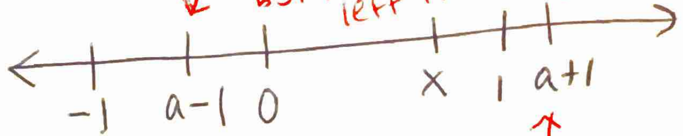
Note that $|x^4 - a^4| = |(x^2+a^2)(x^2-a^2)|$
$$= |x^2+a^2|\,|x-a|\,|x+a|.$$

Suppose $|x-a| < 1$. ← arbitrary starting bound on $|x-a|$

Then $-1 < x-a < 1$.

$a-1$ could be positive or negative but the furthest over to the left it can be is -1 which happens when $a=0$.

So, $a-1 < x < a+1$.



ie $a-1 \geq -1$ since $a \geq 0$

$a+1 \geq 1$ since $a \geq 0$

Thus, $0 \leq |x| < a+1$.

So, $|x+a| \leq |x| + |a|$
$$= |x| + a$$
$$< a+1+a = 2a+1.$$

And $|x^2+a^2| \leq |x^2| + |a^2| = |x|^2 + a^2 < (a+1)^2 + a^2$.

So, if $|x-a| < 1$, then
$$|x^2+a^2|\,|x-a|\,|x+a| < [(a+1)^2+a^2]\,|x-a|\,(2a+1)$$

Let $\delta = \min\left\{1, \dfrac{\varepsilon}{[(a+1)^2+a^2][2a+1]}\right\}$.

If $|x-a| < \delta$, then
$$|x^4-a^4| = |x^2+a^2|\,|x-a|\,|x+a| < [(a+1)^2+a^2](2a+1)\,|x-a|$$
$$< [(a+1)^2+a^2](2a+1)\frac{\varepsilon}{[(a+1)^2+a^2](2a+1)} = \varepsilon.$$

∎

[If $a < 0$, then you can do a similar proof. You just have to be careful here]

④ Let $a \in \mathbb{R}$ with $a \neq 0$. We prove the case where $a > 0$. Try $a < 0$ to finish the proof.

Let $\varepsilon > 0$.

Note that $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{a^2 - x^2}{a^2x^2}\right| \underset{\boxed{|-y|=|y|}}{=} \left|\frac{x^2 - a^2}{a^2x^2}\right| = \frac{|x-a||x+a|}{|a^2||x^2|}$.

Suppose that $\delta \leq \frac{a}{2}$ ← $\boxed{\text{stay away from asymptote}}$

Suppose that $|x-a| < \delta \leq \frac{a}{2}$.

Then $-\frac{a}{2} < x - a < \frac{a}{2}$.

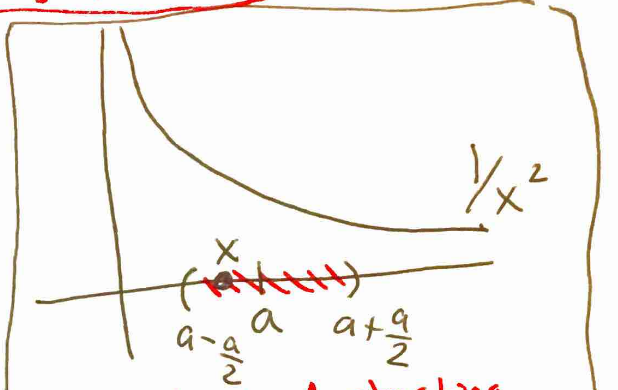
So, $\frac{a}{2} < x < \frac{3a}{2}$.

Thus, $\frac{3a}{2} < x + a < \frac{5a}{2}$.

So, $|x+a| < \frac{5a}{2}$.

Also, $\frac{a^2}{4} < x^2 < \frac{9a^2}{4}$.

So, $\frac{1}{x^2} < \frac{4}{a^2}$.



picture of starting bound on δ

Therefore if $|x-a| < \frac{a}{2}$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x-a||x+a|}{|a^2||x^2|} \underset{\boxed{a^2>0}}{<} \frac{|x-a| \cdot \frac{5a}{2}}{a^2} \cdot \frac{4}{a^2} = |x-a| \cdot \frac{10}{a^3}$$

Let $\delta = \min\left\{\frac{a}{2}, \frac{\varepsilon}{\left(\frac{10}{a^3}\right)}\right\}$.

If $|x-a| < \delta$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \underset{\boxed{|x-a|<\frac{a}{2}}}{<} |x-a| \cdot \frac{10}{a^3} \underset{\boxed{|x-a|<\frac{\varepsilon}{\left(\frac{10}{a^3}\right)}}}{<} \frac{\varepsilon}{\left(\frac{10}{a^3}\right)} \cdot \left(\frac{10}{a^3}\right) = \varepsilon. \quad \square$$

(5) Let $\varepsilon > 0$. Let $a \in \mathbb{R}$ with $a > 0$.

Note that

$$|\sqrt{x}-\sqrt{a}| = \left|\left(\frac{\sqrt{x}-\sqrt{a}}{1}\right)\cdot\left(\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right)\right|$$

$$= \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|}$$

Suppose that $\delta \leq \frac{a}{2}$. $\leftarrow$ $\boxed{\text{stay away from 0}}$

Suppose that $|x-a| < \delta \leq \frac{a}{2}$.

Then $-\frac{a}{2} < x-a < \frac{a}{2}$.

So, $\frac{a}{2} < x < \frac{3a}{2}$.

Thus,

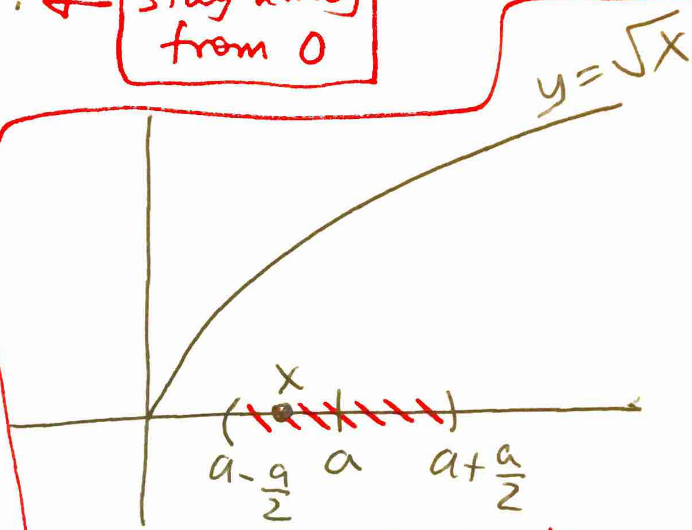
$\sqrt{x}+\sqrt{a} > \sqrt{\frac{a}{2}}+\sqrt{a}$.

So, since $x > 0$, we have

$$\frac{1}{|\sqrt{x}+\sqrt{a}|} = \frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{\frac{a}{2}}+\sqrt{a}}.$$

So, if $|x-a| < \frac{a}{2}$, then

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} < \frac{|x-a|}{\sqrt{\frac{a}{2}}+\sqrt{a}}.$$



picture of starting bound for δ.
Stay away from 0!
Don't want to cross over into negative territory

Let $\delta = \min\left\{\frac{a}{2}, \varepsilon\left(\sqrt{\frac{a}{2}}+\sqrt{a}\right)\right\}$.

If $|x-a| < \delta$, then

$$|\sqrt{x}-\sqrt{a}| < \frac{|x-a|}{\sqrt{\frac{a}{2}}+\sqrt{a}} < \frac{\varepsilon\left(\sqrt{\frac{a}{2}}+\sqrt{a}\right)}{\sqrt{\frac{a}{2}}+\sqrt{a}} = \varepsilon. \quad \boxtimes$$

$\boxed{|x-a| < \frac{a}{2}}$ $\quad$ $\boxed{|x-a| < \varepsilon\left(\sqrt{\frac{a}{2}}+\sqrt{a}\right)}$

(6)

(a) Let $a \in \mathbb{R}$.

Let $\varepsilon > 0$. Pick $\delta = \varepsilon$. Then if $|x-a| < \delta$, then $|x-a| < \varepsilon$. $\blacksquare$

(b) Let $a \in \mathbb{R}$.

Let $\varepsilon > 0$. Pick δ to be any positive real number. If $|x-a| < \delta$, then $|\alpha - \alpha| = 0 < \varepsilon$. $\blacksquare$

(c) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial with $a_i \in \mathbb{R}$ for all i.
Then f is the sum and product of constants and x's. From (a) and (b), constant functions and x are continuous on all of $\mathbb{R}$. From class, products and sums of continuous functions are continuous. Thus, polynomials are continuous on all of $\mathbb{R}$.

(7) Let $\varepsilon > 0$.

(a) Since f is continuous on D and $L \in D$ there exists $\delta > 0$ so that if $x \in D$ and $|x - L| < \delta$, then $|f(x) - f(L)| < \varepsilon$.

Since (a_n) converges to L, there exists $N > 0$ so that if $n \geq N$, then $|a_n - L| < \delta$.
Note that each a_n is in D.
Thus, if $n \geq N$ then $|a_n - L| < \delta$ and $a_n \in D$, so

$$|f(a_n) - f(L)| < \varepsilon.$$

Hence the sequence $(f(a_n))$ converges to $f(L)$. $\blacksquare$

(b) Let $\varepsilon > 0$.
Since f is continuous at L, there exists $\delta_1 > 0$ so that if $y \in D$ and $|y - L| < \delta_1$, then $|f(y) - f(L)| < \varepsilon$.
Since $\lim_{x \to a} g(x) = L$, there exists $\delta_2 > 0$ so that if $0 < |x - a| < \delta_2$, then $|g(x) - L| < \delta_1$.
Note that $g(x) \in D$ for all x, by the problem assumption.
Hence, if $0 < |x - a| < \delta_2$, then $|f(g(x)) - f(L)| < \varepsilon$.

Therefore, $\lim_{x \to a} f(g(x)) = f(L)$. $\blacksquare$