Math 5800
Homework # 4
Step functions

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1. Do the following for each step function below: (i) draw a picture of the step function, (ii) express the step function in a representation that involves only disjoint intervals, and (iii) find the integral of the step function.

   (a) $f = 2 \cdot \chi_{[0,4)} + 3 \cdot \chi_{[1,3)} - 4 \cdot \chi_{(2,4)}$

   (b) $g = -2 \cdot \chi_{[-1,2)} + 5 \cdot \chi_{[1,3]}$

   (c) $h = 2 \cdot \chi_{[-4,4)} + 4\pi \cdot \chi_{(-1,-1)} + -3 \cdot \chi_{[0,0]} + \cdot \chi_{[2,2]}$

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2. Let $S, T \subseteq \mathbb{R}$ with $S \subseteq T$. Prove that $\chi_S(x) \leq \chi_T(x)$ for all $x \in \mathbb{R}$.

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3. Let $S_1, S_2, \ldots, S_n \subseteq \mathbb{R}$. Let $S = \cup_{k=1}^{n} S_k$. Prove that

$$\chi_S(x) \leq \sum_{k=1}^{n} \chi_{S_k}(x)$$

   for all $x \in \mathbb{R}$.

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4. Let $S, A_1, A_2, \ldots, A_r \subseteq \mathbb{R}$. Suppose the A_i are disjoint sets, that is, suppose that $A_k \cap A_j = \emptyset$ if $k \neq j$.

   Prove that $S = A_1 \cup A_2 \cup \ldots \cup A_r$ if and only if

$$\chi_S = \chi_{A_1} + \chi_{A_2} + \ldots + \chi_{A_r}$$

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5. (a) If f is a step function, prove that $|f|$ is a step function. Here $|f|(x) = |f(x)|$.

   (b) Given two step functions χ_1 and χ_2 define a new function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \max\{\chi_1(x), \chi_2(x)\}$ for each $x \in \mathbb{R}$. Prove that f is a step function.

   (c) Given two step functions χ_1 and χ_2 define a new function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \min\{\chi_1(x), \chi_2(x)\}$ for each $x \in \mathbb{R}$. Prove that f is a step function.

Hint: For parts (b) and (c) these formulas will be helpful:

$$max\{\chi_1, \chi_2\} = \frac{1}{2}\,\chi_1 + \frac{1}{2}\,\chi_2 + \frac{1}{2}\,|\chi_1 - \chi_2|$$

and

$$min\{\chi_1, \chi_2\} = \frac{1}{2}\,\chi_1 + \frac{1}{2}\,\chi_2 - \frac{1}{2}\,|\chi_1 - \chi_2|$$

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6. Given a function $f : \mathbb{R} \to \mathbb{R}$, define $f^+ : \mathbb{R} \to \mathbb{R}$ as

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

   (a) Let $g = 2\chi_{[0,2)} - 4\chi_{(-2,-1]} - 3\chi_{[-3,-3]} + \chi_{(3,4]}$.
   Find g^+ and draw a picture of g^+.

   (b) Prove that if f is a step function, then f^+ is a step function.

7. Recall that $\mathscr{R}$ denotes the set of all subsets S of $\mathbb{R}$ such that S can be written as $S = I_1 \cup I_2 \cup \ldots \cup I_r$ where $I_1, I_2, \ldots, I_r$ are disjoint bounded intervals.

   (a) $S \in \mathscr{R}$ if and only if χ_S is a step function.

   (b) If $S, T \in \mathscr{R}$, then $S \cup T, S \cap T, S - T$ are all in $\mathscr{R}$.

   (c) If $S, T \in \mathscr{R}$ and $S \subseteq T$, then $\ell(S) \leq \ell(T)$

   (d) If $A = \bigcup_{i=1}^{s} A_i$ where each A_i is a bounded interval, then $A \in \mathscr{R}$

   (e) Let $I_1, I_2, \ldots, I_r$ be disjoint bounded intervals. Suppose that there exist bounded intervals $J_1, J_2, \ldots, J_t$ where $\displaystyle\bigcup_{j=1}^{r} I_j \subseteq \bigcup_{i=1}^{t} J_i$. Prove that $\displaystyle\sum_{j=1}^{r} \ell(I_j) \leq \sum_{i=1}^{t} \ell(J_i)$.

   Hint for (b):

$$\chi_{S \cup T} = max\{\chi_S, \chi_T\} \quad \chi_{S \cap T} = min\{\chi_S, \chi_T\} \quad \chi_{S-T} = (\chi_S - \chi_T)^+$$