Math 465 - Homework # 4
Continuity

1. Use the ϵ-δ definition of continuity to prove that $f(x) = x^2 - 1$ is continuous at $a = 2$.

2. Use the ϵ-δ definition of continuity to prove that $f(x) = 3x^2 + 1$ is continuous at $a = 1$.

3. Use the ϵ-δ definition of continuity to prove that $f(x) = x^4$ is continuous at every real number $a \geq 0$.

4. Use the ϵ-δ definition of continuity to prove that $f(x) = \dfrac{1}{x^2}$ is continuous at every $a \in \mathbb{R}$ with $a \neq 0$.

5. Use the ϵ-δ definition of continuity to prove that $f(x) = \sqrt{x}$ is continuous at every $a \in \mathbb{R}$ with $a > 0$.

6. Prove the following.

   (a) Prove that $f(x) = x$ is continuous on all of $\mathbb{R}$.

   (b) Let α be a constant real number. Prove that the constant function $f(x) = \alpha$ is continuous for all of $\mathbb{R}$.

   (c) Prove that polynomials are continous on all of $\mathbb{R}$.

7. (This problem shows how you can pull a limit inside of a continuous function.) Let $f : D \to \mathbb{R}$ be a continuous function where D is a subset of $\mathbb{R}$.

   (a) Suppose that (a_n) is a sequence of real numbers with $\lim\limits_{n\to\infty} a_n = L$ where $a_n \in D$ for all $n \in \mathbb{N}$ and $L \in D$. Prove that

   $$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

   (b) Suppose that $g : A \to \mathbb{R}$ where A is a subset of $\mathbb{R}$ and suppose that the range of g is contained in D. Suppose that $a \in A$ and $\lim\limits_{x\to a} g(x) = L$ with $L \in D$. Prove that

   $$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$