($\Longleftarrow$) Suppose that $[T]_\beta$ is diagonalizable.

There exists an invertible matrix Q where $Q^{-1}[T]_\beta Q = D$

and $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$.

Let $\beta = [v_1, v_2, \dots, v_n]$.

ble.

Let $Q = \left(c_1 \middle| c_2 \middle| \cdots \middle| c_n \right)$

where c_i is the ith column of Q.

Then $Q^{-1}[T]_\beta Q = D$ becomes

$[T]_\beta Q = QD$ which is

$$[T]_\beta \left(c_1 \middle| c_2 \middle| \cdots \middle| c_n \right) = \left(c_1 \middle| c_2 \middle| \cdots \middle| c_n \right) \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

So,

$$\left([T]_\beta c_1 \middle| [T]_\beta c_2 \middle| \cdots \middle| [T]_\beta c_n \right) = \left(\lambda_1 c_1 \middle| \lambda_2 c_2 \middle| \cdots \middle| \lambda_n c_n \right)$$

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$A = ($ 1 2

$A ($ 1 3

SIDE EXAMPLE

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

$$A \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} (1,-1)\begin{pmatrix} 1 \\ 3 \end{pmatrix} & (1,-1)\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ (2,3)\begin{pmatrix} 1 \\ 3 \end{pmatrix} & (2,3)\begin{pmatrix} 0 \\ 2 \end{pmatrix} \end{pmatrix}$$

$$= \left(A\begin{pmatrix} 1 \\ 3 \end{pmatrix} \,\middle|\, A\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right)$$

$$\boxed{\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -9 & 20 \end{pmatrix} = \left(-3\begin{pmatrix} 1 \\ 3 \end{pmatrix} \,\middle|\, 10\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right)}$$

So, $[T]_\beta c_1 = \lambda_1 c_1$

$$[T]_\beta c_2 = \lambda_2 c_2$$

$$\vdots \qquad \vdots$$

$$[T]_\beta c_n = \lambda_n c_n \, .$$

Since Q is invertible, none of its columns can be $\vec{0}$.

So, $c_i \neq \vec{0}$.

So, each c_i is an eigenvector of $[T]_\beta$ with eigenvalue λ_i.

Suppose $c_j = \begin{pmatrix} q_{1j} \\ q_{2j} \\ \vdots \\ q_{nj} \end{pmatrix}$ where $q_{ij} \in F$.

Define $m_1, m_2, \ldots, m_n \in V$ where $[m_j]_\beta = c_j$ that is

$$m_j = q_{1j} v_1 + q_{2j} v_2 + \cdots + q_{nj} v_n$$

(Recall: $\beta = (v_1, v_2, \ldots, v_n)$)

We also have $[T(m_j)]_\beta = [T]_\beta [m_j]_\beta = [T]_\beta c_j = \lambda_j c_j = \lambda_j [m_j]_\beta$

Thm

def of m_j

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So, $[T(m_j)]_\beta = \lambda_j \begin{pmatrix} q_{1j} \\ q_{2j} \\ \vdots \\ q_{nj} \end{pmatrix} = \begin{pmatrix} \lambda_j q_{1j} \\ \lambda_j q_{2j} \\ \vdots \\ \lambda_j q_{nj} \end{pmatrix}.$

Thus, $T(m_j) = (\lambda_j q_{1j}) v_1 + (\lambda_j q_{2j}) v_2 + \cdots + (\lambda_j q_{nj}) v_n = \lambda_j (q_{1j} v_1 + q_{2j} v_2 + \cdots + q_{nj} v_n) = \lambda_j m_j$

Since $c_j \neq \vec{0}$, we must $m_j \neq \vec{0}$ $\forall j$.

So, m_j is an eigenvector of T with eigenvalue λ_j

If we can show that $m_1, m_2, \dots, m_n$ are lin. ind. then they are a basis of eigenvectors for T and then T is diagonalizable.

Suppose

$$\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n = \vec{0}$$

Then

$$\begin{aligned}
&\alpha_1(q_{11}v_1 + q_{21}v_2 + \cdots + q_{n1}v_n) + \\
&\alpha_2(q_{12}v_1 + q_{22}v_2 + \cdots + q_{n2}v_n) + \\
&\quad \vdots \qquad\qquad \vdots \\
&\alpha_n(q_{1n}v_1 + q_{2n}v_2 + \cdots + q_{nn}v_n) = \vec{0}
\end{aligned}$$

So,

Sin

So, $(\alpha_1 q_{11} + \alpha_2 q_{12} + \cdots + \alpha_n q_{1n}) v_1 + (\alpha_1 q_{21} + \alpha_2 q_{22} + \cdots + \alpha_n q_{2n}) v_2 + \cdots + (\alpha_1 q_{n1} + \alpha_2 q_{n2} + \cdots + \alpha_n q_{nn}) v_n = \vec{0}$.

Since $\beta = (v_1, v_2, \ldots, v_n)$ is lin. ind,

$$\begin{aligned}
\alpha_1 q_{11} + \alpha_2 q_{12} + \cdots + \alpha_n q_{1n} &= 0 \\
\alpha_1 q_{21} + \alpha_2 q_{22} + \cdots + \alpha_n q_{2n} &= 0 \\
&\vdots \\
\alpha_1 q_{n1} + \alpha_2 q_{n2} + \cdots + \alpha_n q_{nn} &= 0
\end{aligned}$$

Thus,

$$\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, $Q \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Thus, $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

So, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

So, $m_1, m_2, \ldots, m_n$ are lin. ind. $\blacksquare$