Homework 5 Solutions

(1) Let $x \in \mathbb{R}$. Then every interval (a,b) containing x is also contained in $\mathbb{R}$. Hence, x is an interior point. So, $\mathbb{R}$ is open.

~~(2) The statement:~~

~~"If $x \in \emptyset$, then x is an interior point" is true for all x since there are no such x.~~

~~Hence every point in $\emptyset$~~

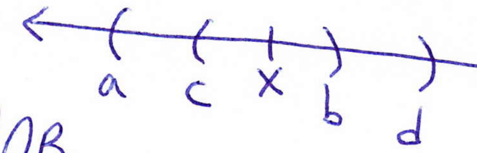
(2) Every point $x \in \emptyset$ is an interior point since there are no points in $\emptyset$.

(3) Suppose that A and B are open subsets of $\mathbb{R}$.

(a) Let $x \in A \cap B$. Since $x \in A$ and A is open there exists an interval (a,b) with $x \in (a,b) \subseteq A$. Similarly there exists an interval (c,d) with $x \in (c,d) \subseteq B$.

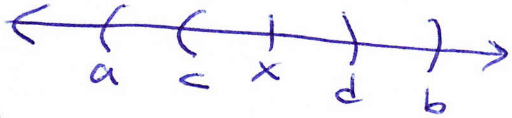
There are a few cases to consider:

Case 1: Suppose $a \leq c \leq b \leq d$.



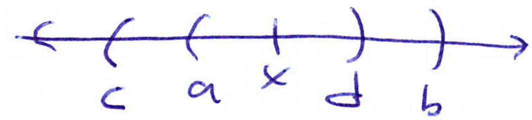
Then, $x \in (a,b) \cap (c,d) = (c,b) \subseteq A \cap B$.

Case 2: Suppose $a \leq c \leq d \leq b$.



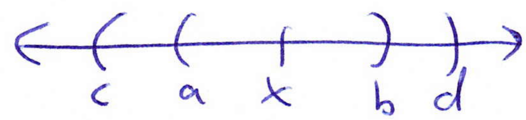
Then $x \in (a,b) \cap (c,d) = (c,d) \subseteq A \cap B$

Case 3: Suppose $c \leq a \leq d \leq b$.



Then $x \in (a,b) \cap (c,d) = (a,d) \subseteq A \cap B$.

Case 4: Suppose $c \leq a \leq b \leq d$.



Then $x \in (a,b) \cap (c,d) = (a,b) \subseteq A \cap B$.

In either case x is an interior point of $A \cap B$.
Thus, $A \cap B$ is open.

(b) Let $x \in A \cup B$.
Then $x \in A$ or $x \in B$.
1: Suppose $x \in A$.
Since A is open, x is an interior point.
Thus there exists an interval (a,b) with $x \in (a,b) \subseteq A$.
So, $x \in (a,b) \subseteq A \cup B$.
Hence x is an interior point of $A \cup B$.
2: The same thing happens if $x \in B$.
Thus, $A \cup B$ is open

(4) (a)/(b) Suppose that A and B are closed subsets of $\mathbb{R}$.

Then $\mathbb{R}\setminus A$ and $\mathbb{R}\setminus B$ are open.

Hence,

$$\mathbb{R}\setminus(A\cap B) \overset{\boxed{\text{DeMorgan}}}{=} (\mathbb{R}\setminus A)\cup(\mathbb{R}\setminus B) \text{ is open by } 3(b).$$

and

$$\mathbb{R}\setminus(A\cup B) = (\mathbb{R}\setminus A)\cap(\mathbb{R}\setminus B) \text{ is open by } 3(a).$$

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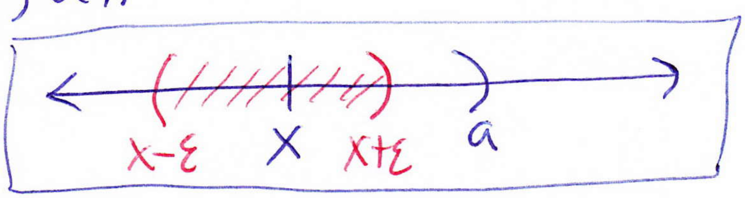
(5) (a) Let $x \in (-\infty, a)$.

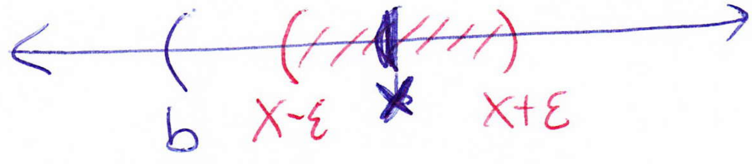
Let $\varepsilon = \frac{a-x}{2}$.



Then $x \in (x-\varepsilon, x+\varepsilon) \subseteq (-\infty, a)$.
So, x is an interior point. Thus, $(-\infty, a)$ is open.

(b) Let $x \in (b, \infty)$.

Let $\varepsilon = \frac{x-b}{2}$.



Then, $x \in (x-\varepsilon, x+\varepsilon) \subseteq (b, \infty)$.
So, x is an interior point.
Thus, (b, ∞) is open.

(c) $\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$

which is open by ③(b), ⑤(a), and ⑤(b).

Thus, $[a,b]$ is closed.

(d) $\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is open by ⑤(a).

Thus, $[a, \infty)$ is closed.

(e). $\mathbb{R} \setminus (-\infty, b] = (b, \infty)$ is open by ⑤(b).

Hence $(-\infty, b]$ is closed.

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⑥ Without loss of generality, assume that $x_1 < x_2 < \dots < x_n$. (Just reorder the elements of S otherwise.)

(a) $\mathbb{R} \setminus S = (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup \dots \cup (x_{n-1}, x_n) \cup (x_n, \infty)$.

By ⑤(a), ⑤(b), the fact that (a,b) is open for any $a<b$, and the repeated application of ③(b), we get that $\mathbb{R} \setminus S$ is open. Thus, S is closed.

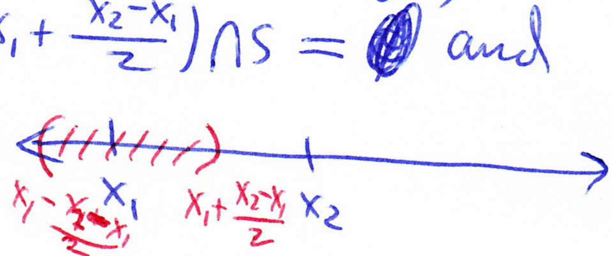
(b) Let $x \in \mathbb{R}$.

We show that x is not a limit point of S.

<u>case 1: $x = x_i$ for some i.</u>

If $x = x_1$, then $\left(x_1 - \frac{x_2 - x_1}{2}, x_1 + \frac{x_2 - x_1}{2}\right) \cap S = \{x_1\}$ and

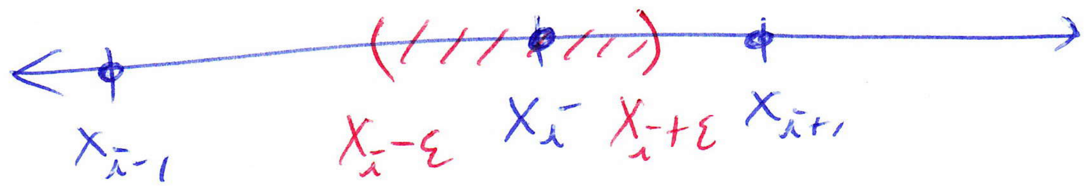
$x \in \left(x_1 - \frac{x_2 - x_1}{2}, x_1 + \frac{x_2 - x_1}{2}\right)$



If $x = x_i$ with $2 \le i \le n-1$, then set

$$\varepsilon = \min\left\{ \frac{x_i - x_{i-1}}{2}, \frac{x_{i+1} - x_i}{2} \right\}.$$

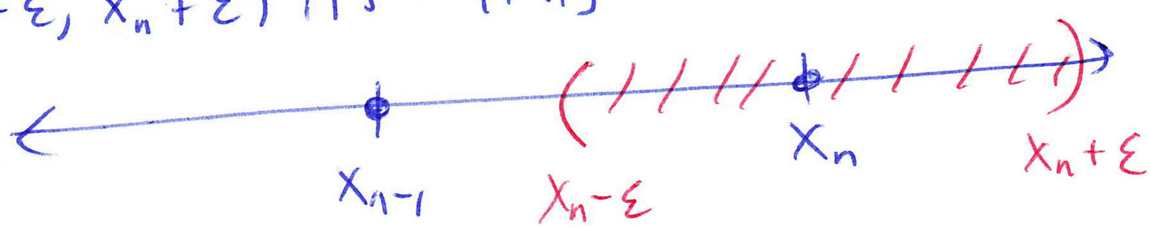
Then $x_i \in (x_i - \varepsilon, x_i + \varepsilon)$ and $(x_i - \varepsilon, x_i + \varepsilon) \cap S = \{x_i\}$



If $x = x_n$, set $\varepsilon = \dfrac{x_n - x_{n-1}}{2}$.

Then $x_n \in (x_n - \varepsilon, x_n + \varepsilon)$ and

$(x_n - \varepsilon, x_n + \varepsilon) \cap S = \{x_n\}$

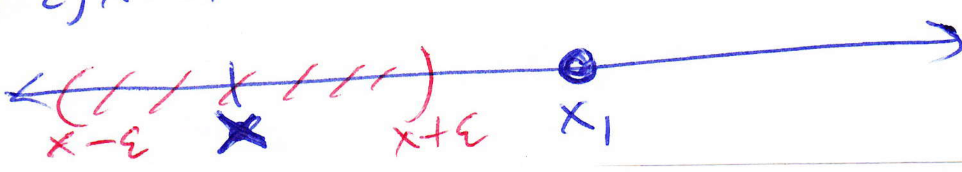


In all cases about we have put an interval around x containing no points of S except for x.

case 2: $x \ne x_i$ for all $1 \le i \le n$

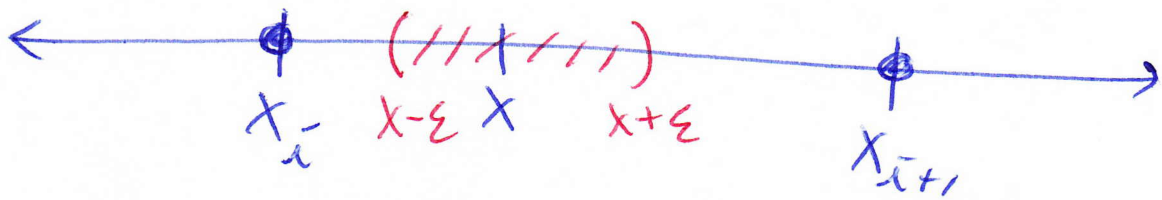
If $x < x_1$, set $\varepsilon = \dfrac{x_1 - x}{2}$.

Then $(x - \varepsilon, x + \varepsilon) \cap S = \phi$ and $x \in (x - \varepsilon, x + \varepsilon)$



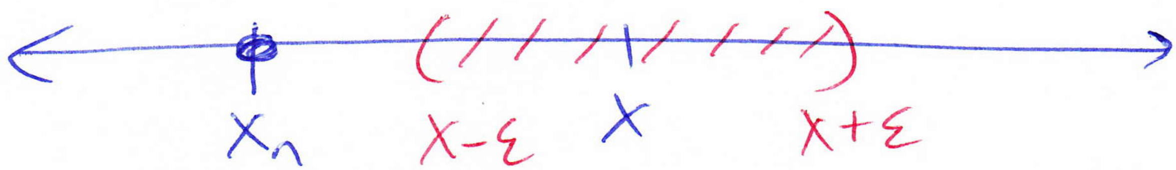
If $x_i < x < x_{i+1}$ for some i, then set $\varepsilon = \min\left\{ \frac{x - x_i}{2}, \frac{x_{i+1} - x}{2} \right\}$.

Then $x \in (x-\varepsilon, x+\varepsilon)$ and $(x-\varepsilon, x+\varepsilon) \cap S = \emptyset$,



If $x_n < x$, set $\varepsilon = \frac{x - x_n}{2}$.

Then $x \in (x-\varepsilon, x+\varepsilon)$ and $(x-\varepsilon, x+\varepsilon) \cap S = \emptyset$,



In all cases, we have found an interval around x containing no points of S.

Therefore, by cases 1 & 2, x is <u>not</u> a limit point of S.

(7) $A = (0,1)$

$x = 1$

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(8) Let $A_n = (1-\frac{1}{n}, 1+\frac{1}{n})$.

Then $A_1 = (0, 2)$

$A_2 = (1-\frac{1}{2}, 1+\frac{1}{2})$

$A_3 = (1-\frac{1}{3}, 1+\frac{1}{3})$

$\vdots \quad \vdots$

Each A_n is open, and $\bigcap_{n=1}^{\infty} A_n = \{1\}$

which is not open. because $1 \in \bigcap_{n=1}^{\infty} A_n$

but 1 is not an interior point of $\bigcap_{n=1}^{\infty} A_n$.

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(9) Let $B_n = [1+\frac{1}{n}, 2-\frac{1}{n}]$, with $n \geq 2$.

Then each B_n is closed, and $\bigcup_{n=2}^{\infty} B_n = (1, 2)$ which is not closed because its complement $\mathbb{R} \setminus \bigcup_{n=2}^{\infty} B_n = (-\infty, 1] \cup [2, \infty)$ is not open (because $1, 2 \in \mathbb{R} \setminus \bigcup_{n=2}^{\infty} B_n$ but are not interior points).

$B_2 = [1+\frac{1}{2}, 2-\frac{1}{2}]$

$B_3 = (1+\frac{1}{3}, 2-\frac{1}{3}]$

$\vdots \quad \vdots$

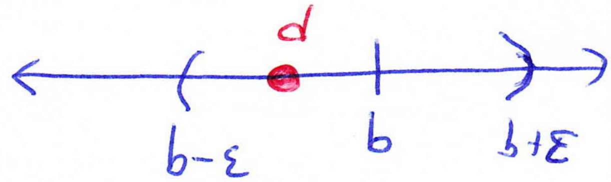
(10) Since D is bounded, D is contained in some interval $[-M, M]$. Thus, D is bounded from below and from above. By the completeness axiom, $a = \inf(D)$ and $b = \sup(D)$ exist.

We now show that $b \in D$. A similar proof will show that $a \in D$.

Suppose to the contrary that $b \notin D$.

Let $\varepsilon > 0$. Since $b = \sup(D)$, [by sup useful fact] there exists $d \in D$ with $b - \varepsilon < d < b$.

[$d \neq b$ since $b \notin D$]



This shows that b is a limit point of D. Since D is closed, $b \in D$. Contradiction!