Math 5800
Homework # 5
More 4650 Review

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1. Let $S \subseteq \mathbb{R}$ with $S \neq \emptyset$. Let $b \in \mathbb{R}$. Prove that b is the infimum of S if and only if the following two conditions hold:

   (i) b is a lower bound for S, and

   (ii) for every $\epsilon > 0$ there exists $x \in S$ with $b \leq x < b + \epsilon$

   Hint: To prove the above use the fact that b is the supremum of S if and only if (i) b is a lower bound for S and (ii) if c is any other lower bound for S then $c \leq b$ (ie, b is the greatest lower bound for S).

   If you want more practice, you could prove the similar statement from class about the supremum of a set S.

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2. (Monotone Convergence Theorem) Prove: If $(a_n)_{n=1}^{\infty}$ is a non-decreasing sequence that is bounded from above, then $(a_n)_{n=1}^{\infty}$ converges.

   Recall: $(a_n)_{n=1}^{\infty}$ is bounded from above if there exists $M > 0$ such that $a_n \leq M$ for all $n \geq 1$.

   Hint: To do this problem, first demonstrate that

   $$S = \{a_k \mid k = 1, 2, 3, 4, \ldots\} = \{a_1, a_2, a_3, \ldots\}$$

   is bounded from above. Then prove that

   $$\lim_{n \to \infty} a_n = sup(S)$$

   To do this use the property that b is the supremum of S if (i) b is an upper bound for S, and (ii) for every $\epsilon > 0$ there exists $x \in S$ with $b - \epsilon < x \leq b$.

   Note: If you want more practice, then you could prove the similar statement for bounded non-increasing sequences that we stated in class.
