Math 465 - Homework # 6
Compact Sets

1. Let $x_1, x_2, \ldots, x_n$ be real numbers. Show that the finite set $\{x_1, x_2, \ldots, x_n\}$ is compact using the definition of compactness.

2. Consider the set $S = [1, \infty)$. Consider the open cover

$$X = \{(n-1, n+1) \mid n \in \mathbb{N}\} = \{(0,2), (1,3), (2,4), (3,5), \ldots\}$$

of S. Prove that X contains no finite subcover of S. Hence S is not compact.

3. Consider the set $S = (-1, 1)$ Consider the open cover

$$X = \left\{ \left(x - \frac{1}{4}, x + \frac{1}{4} \right) \mid x \in \mathbb{Q} \text{ and } -1 < x < 1 \right\}.$$

Find a finite subcover of X that covers S. (Note that although X contains a finite subcover of S, S is not compact—because S is not closed. So, there must exist an infinite cover of S that has no finite subcover—even though the cover X given above does not.)

4. Let $f : [a, \infty) \to \mathbb{R}$ be continous on all of $[a, \infty)$. Suppose that $\lim_{x \to \infty} f(x)$ exists. Prove that f is bounded on $[a, \infty)$.

5. Let A and B be compact subsets of $\mathbb{R}$.

   (a) Prove that $A \cap B$ is compact.

   (b) Prove that $A \cup B$ is compact.

   (c) Find an infinite family A_n of compact sets for which $\cup_{n=1}^{\infty} A_n$ is not compact.

   (d) Suppose that A_n is a compact set for $n \geq 1$. Prove that $\cap_{n=1}^{\infty} A_n$ is compact.

6. Is S compact or not compact? Explain why.

   (a) $S = (0, 1]$

   (b) $S = [-10, 0] \cup [1, 2]$

(c) $S = (-\infty, 2)$

(d) $S = (-\infty, 10]$

(e) $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$

7. Suppose that (a_n) is a sequence that converges to L. Prove that the set

$$A = \{a_n | n \in \mathbb{N}\} \cup \{L\}$$

is compact.