Math 446 - Homework # 6

1. Do the following calculations in $\mathbb{Z}[i]$.
   (a) $(2 + 10i) + (-3 + 15i)$
   (b) $(-13 + i) - (2 - 3i)$
   (c) $(1 + 3i)(2 - 10i)$
   (d) $\frac{1+i}{i}$
   (e) $\frac{2-3i}{1-2i}$
2. Calculate the norms of the following elements of $\mathbb{Z}[i]$.
   (a) i
   (b) $2 - i$
   (c) 15
   (d) $15 + 102i$
3. List all the associates of $-1 + 2i$.
4. List all the associates of 10.
5. Carry out the division algorithm for z and w. That is, find q and r in $\mathbb{Z}[i]$ with $z = wq + r$.
   (a) $z = -8 - i$ and $w = 3 + 2i$
   (b) $z = 5 + i$ and $w = -1 - 2i$
   (c) $z = 33 + 5i$ and $w = 10 - 2i$
6. Determine whether or not $2 + 3i$ divides $10 - 11i$ in $\mathbb{Z}[i]$.
7. Determine whether or not $3 - 2i$ divides $10 + i$ in $\mathbb{Z}[i]$.
8. Determine whether or not $2 + i$ is prime in $\mathbb{Z}[i]$. Find all the divisors of $2 + i$.
9. Let w and v be Gaussian integers with $w \neq 0$ and $v \neq 0$. If w divides v and $N(w) = N(v)$, then w is an associate of v.

10. Can there exist Gaussian integers z and w where $N(z)$ divides $N(w)$, but z does not divide w ? Try to find some cases that are non-trivial, ie where $1 < N(z) < N(w)$. [Hint: You might need to write a computer program.]

11. Determine whether or not 2 is prime in $\mathbb{Z}[i]$. Find all the divisors of 2.

12. Determine whether or not 13 is prime in $\mathbb{Z}[i]$. Find all the divisors of 13.

13. Let z be a Gaussian integer. Suppose that z is not prime in $\mathbb{Z}[i]$. Suppose further that $z \neq 0$ and z is not a unit. Then there exist Gaussian integers w and v where

    (a) $z = wv$

    (b) w is not a unit and w is not an associate of z

    (c) v is not a unit and v is not an associate of z

    That is, z factors non-trivially.

14. Let p be an odd prime in $\mathbb{Z}$ with $p \equiv 1(\text{mod } 4)$. Prove that p is not prime in $\mathbb{Z}[i]$.

15. Let p be an odd prime in $\mathbb{Z}$ with $p \equiv 3(\text{mod } 4)$. Prove that p is prime in $\mathbb{Z}[i]$.

16. Let $z, w \in \mathbb{Z}[i]$. Prove that w divides z if and only if $\overline{w}$ divides $\overline{z}$.

17. (a) $N(v) = N(\overline{v})$ for all Gaussian integers v.

    (b) For any Gaussian integer u we have the following: u is a unit iff $\overline{u}$ is a unit.

    (c) Let $z \in \mathbb{Z}[i]$. Prove that z is prime if and only if $\overline{z}$ is prime.

18. Let $z \in \mathbb{Z}[i]$. Prove that if $N(z)$ is a prime in $\mathbb{Z}$, then z is prime in $\mathbb{Z}[i]$.

19. Let $w, y, z \in \mathbb{Z}[i]$. Prove that if w is a unit and z divides wy, then z divides y.