Math 5800
Homework # 7
The Lebesgue integral

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1. (a) If f is a step function, then $f \in L^0$.
   (b) If f is a step function, then $f \in L^1$.

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2. Let
$$f = \chi_{\mathbb{R}}$$
   (a) Show that $f \notin L^1$.
   (b) Show that $f \in L^1(I)$ for any finite interval I.

   [*Hint for (a): Define*
$$g_k(x) = \begin{cases} 1 & \text{if } x \in [-k, k] \\ 0 & \text{if } x \notin [-k, k] \end{cases}$$
   Show that g_k is in L^1 and that $\int g_k = 2k$ for all $k \geq 1$. Show that $g_k(x) \leq f(x)$ for all x. Conclude that if $f \in L^1$ then $\int g_k \leq \int f$ for all $k \geq 1$. This will lead to a contradiction.]

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3. Let $f, g \in L^0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$ and $\beta \geq 0$.
   - Prove that $\alpha f + \beta g \in L^0$.
   - Prove that $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$.

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4. Let $f \in L^0$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose also that $f(x) = g(x)$ for almost all x in $\mathbb{R}$.
   - Prove that $g \in L^0$.
   - Prove that $\int g = \int f$.

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5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and that $(\phi_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions that converges almost everywhere to f. Suppose also that there exists a real number $M > 0$ where the sequence $\int \phi_n \leq M$ for all $n \geq 1$.

   (a) Prove that $f \in L^0$.

   (b) Prove that $\displaystyle\int f = \lim_{n\to\infty} \int \phi_n$.

   (c) Prove that $\int f \leq M$.

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6. Suppose that $a \leq c \leq b$. If $f \in L^1([a,c])$ and $f \in L^1([c,b])$, then $f \in L^1([a,b])$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

[Hint: Show that

$$f \cdot \chi_{[a,b]} = f \cdot \chi_{[a,c]} + f \cdot \chi_{[c,b]}$$

almost everywhere]

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7. Suppose that f is integrable on the interval $[a,b]$ and that there are real numbers m, M such that

$$m \leq f(x) \leq M$$

for all $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

[Hint: Show and use this: $m \cdot \chi_{[a,b]} \leq f \cdot \chi_{[a,b]} \leq M \cdot \chi_{[a,b]}$ *]*

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8. (Standard construction problem) Let

$$f(x) = \begin{cases} x+1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Consider the standard construction $(\gamma_n)_{n=1}^{\infty}$ for f on $[-1,1]$. In the homework on sequences of functions and the standard construction, we showed that γ_n converges pointwise to f on all of $\mathbb{R}$.

(a) Use the formula

$$1 + 2 + \cdots + m = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$

to show that

$$\int \gamma_n = \frac{2^n - 1}{2^{n-1}}$$

(b) Show that $f \in L^1$ and that $\int f = 2$.

(c) Conclude that $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x+1$ satisfies $g \in L^1([-1,1])$ and $\int_{-1}^{1}(x+1)dx = 2$.

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9. (Standard construction problem) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Consider the standard construction $(\gamma_n)_{n=1}^{\infty}$ for f on $[0,1]$. In the homework on sequences of functions and the standard construction, we showed that γ_n converges pointwise to f on all of $\mathbb{R}$.

(a) Use the formula

$$1 + 2 + \cdots + m^2 = \sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$$

to show that

$$\int \gamma_n = \frac{2 \cdot 2^{2n} - 3 \cdot 2^n + 1}{6 \cdot 2^{2n}}$$

(b) Show that $f \in L^1$ and that $\int f = 1/3$.

(c) Conclude that $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$ satisfies $g \in L^1([0,1])$ and $\int_0^1 x^2 dx = 1/3$.

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10. Let

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that $g \in L^1(I)$ for any bounded interval I and that

$$\int_I g = \ell(I)$$

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11. Recall the following example from class. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in [0,1] \text{ and } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

Let $\{r_1, r_2, r_3, r_4, \ldots\}$ be an enumeration of the rational numbers that lie inside of $[0,1]$. [For example, it could be something like $\{1/2, 0, 3/10, 51/541, \ldots\}$ but it doesn't have to be this.]

Let $g_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise} \end{cases}$$

Prove the following:

(a) Draw a picture of g_1, g_2, g_3 for a general choice of r_1, r_2, r_3.

(b) $(g_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions

(c) $g_n \to g$ pointwise on all of $\mathbb{R}$

(d) $\int g_n = 0$ for all $n \geq 1$

(e) $g \in L^0$ and $\int g = 0$

This problem was used in a lemma that was proved in class.

12. Let $T_1, T_2, \ldots, T_s$ be disjoint bounded intervals. If there exists $a < b$ where $\displaystyle\bigcup_{i=1}^{s} T_i \subseteq [a, b]$, then $\displaystyle\sum_{i=1}^{s} \ell(T_i) \leq b - a$.