Math 455

Homework # 8 - Homomorphisms and the Kernel

1. For the following functions ϕ, prove that ϕ is a homomorphism. Then find $\text{Ker}(\phi)$ and the image of ϕ.

   (a) Let $\phi : \mathbb{Z} \to \mathbb{Z}$ with $\phi(n) = 5n$.

   (b) Let $\phi : \mathbb{R} \to \mathbb{R}^\times$ with $\phi(x) = 2^x$.

   (c) Let G be an abelian group. Let $\phi : G \to G$ with $\phi(g) = g^{-1}$.

2. Let $\phi : \mathbb{Z} \to \mathbb{Z}_4$ be the homomorphism with $\phi(1) = \overline{2}$. Calculate $\phi(3)$ and $\phi(-2)$. Calculate $\text{Ker}(\phi)$. Calculate $\phi(\mathbb{Z})$.

3. Let $\phi : \mathbb{Z}_8 \to \mathbb{Z}_4$ be the homomorphism with $\phi(\overline{1}) = \overline{3}$. Draw a picture of ϕ. Calculate $\text{Ker}(\phi)$. Calculate $\phi(\mathbb{Z}_8)$.

For the following exercises: Let G and G' be groups. Let e' be the identity of G'. The homomorphism $\phi : G \to G'$ defined by $\phi(g) = e'$ for all $g \in G$ is called the **trivial** homomorphism. Any other homomorphism is called non-trivial.

4. Does there exist a non-trivial homomorphism $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_5$?

5. Does there exist a non-trivial homomorphism $\phi : \mathbb{Z}_3 \to \mathbb{Z}$?

6. Let $\phi : G \to G'$ be a homomorphism. Prove that if $|G|$ is prime, then either ϕ is the trivial homomorphism or ϕ is one-to-one.

7. Let $\phi : G \to G'$ be a homomorphism. Prove that $\phi(G)$ is abelian if and only if $xyx^{-1}y^{-1} \in \text{Ker}(\phi)$ for all $x, y \in G$.