Math 5800

Homework # 9

Measurable Functions

---

1. (a) Let $f(x) = x^2$ for all x. Let $g = \chi_{[-1,2)}$.
   Draw a picture of f, g, and $-g$.
   Draw a picture of $\text{mid}\{-g, f, g\}$.

   (b) Let $f(x) = x^2$ for all x. Let $g = \chi_{[-3,-1)}$.
   Draw a picture of f, g, and $-g$.
   Draw a picture of $\text{mid}\{-g, f, g\}$.

   (c) Let $f(x) = 2x$ for all x. Let $g_n = n \cdot \chi_{[-n,n]}$ for $n \geq 1$. Draw a picture of $f_1 = \text{mid}\{-g_1, f, g_1\}$ and $f_2 = \text{mid}\{-g_2, f, g_2\}$.

   (d) Let

$$f(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ x & \text{if } 0 < x \end{cases}$$

   Let $g_n = n \cdot \chi_{[-n,n]}$ for $n \geq 1$.
   Draw a picture of $f_1 = \text{mid}\{-g_1, f, g_1\}$, $f_2 = \text{mid}\{-g_2, f, g_2\}$, $f_3 = \text{mid}\{-g_3, f, g_3\}$, and $f_4 = \text{mid}\{-g_4, f, g_4\}$

---

2. Let $f : \mathbb{R} \to \mathbb{R}$. For $n \geq 1$ define $g_n = n \cdot \chi_{[-n,n]}$. Let $f_n = \text{mid}\{-g_n, f, g_n\}$. Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$

---

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that g is a non-negative function, that is $g \geq 0$. Let $h = \text{mid}\{-g, f, g\}$. Prove that $|h| \leq g$. [That is, prove that $|h(x)| \leq g(x)$ for all $x \in \mathbb{R}$.]

---

4. Recall that given two functions $h, k : \mathbb{R} \to \mathbb{R}$, define $\max\{h, k\} : \mathbb{R} \to \mathbb{R}$ and $\min\{h, k\} : \mathbb{R} \to \mathbb{R}$ by

$$(\max\{h, k\})(x) = \max\{h(x), k(x)\}$$

and

$$(\min\{h, k\})(x) = \min\{h(x), k(x)\}$$

(a) If $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ are non-decreasing sequences of step functions prove that the sequence $(\min\{\phi_n, \psi_n\})_{n=1}^{\infty}$ is a non-decreasing sequence of step functions.

(b) If $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ are non-decreasing sequences of step functions prove that the sequence $(\max\{\phi_n, \psi_n\})_{n=1}^{\infty}$ is a non-decreasing sequence of step functions.

---

5. Recall the definition of $\min\{h, k\}$ and $\max\{h, k\}$ given above.

(a) Let f and g be in L^0. Prove that $\min\{f, g\}$ and $\max\{f, g\}$ are in L^0.

(b) Let f be in L^1. Then $|f|$ is in L^1.

(c) Let f and g be in L^1. Prove that $\min\{f, g\}$ and $\max\{f, g\}$ are in L^1.

[

Hint for (a): Use the result from the previous exercise.

Hint for (b): If $f = g - h$ where $g \in L^0$ and $h \in L^0$ show that $|f| = max\{g, h\} - min\{g, h\}$.

Hint for (c): Use the fact that

$$min\{h, k\} = \frac{1}{2}h + \frac{1}{2}k - \frac{1}{2}|h - k|$$

and

$$max\{h, k\} = \frac{1}{2}h + \frac{1}{2}k + \frac{1}{2}|h - k|$$

]

6. Let $a, b \in \mathbb{R}$ with $b \geq 0$. Prove that

$$\text{mid}\{-b, a, b\} = \max\{-b, \min\{a, b\}\} = \begin{cases} -b & \text{if } a < -b \\ a & \text{if } -b \leq a \leq b \\ b & \text{if } b < a \end{cases}$$

[*Hint: Break it into three cases:*

$a < -b \leq b$, $-b \leq a \leq b$, *and* $-b \leq b < a$.]

---

7. (a) Let $a, b \in \mathbb{R}$. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers where $b_n \geq 0$ for all n. Prove that if $a_n \to a$ and $b_n \to b$, then $\text{mid}\{-b_n, a_n, b_n\} \to \text{mid}\{-b, a, b\}$.

   (b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions with $f_n : \mathbb{R} \to \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ where g is a non-negative function. Prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all x, then $\lim_{n\to\infty} \text{mid}\{-g, f_n, g\}(x) = \text{mid}\{-g, f, g\}(x)$ for almost all x.

---

8. Let f and h be measurable functions. Let $\alpha \in \mathbb{R}$. Prove the following.

   (a) $f + h$ is measurable

   (b) $\alpha \cdot f$ is measurable.

   (c) $\min\{f, h\}$ is measurable.

   (d) $\max\{f, h\}$ is measurable.

   (e) Let g be a non-negative function in L^1. Suppose that $|f(x)| \leq g(x)$ for almost all x. Prove that f is in L^1.

9. Let $f : \mathbb{R} \to \mathbb{R}$. Let

$$E = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}.$$

Suppose that E has measure zero. Further suppose that f is bounded on any interval $[a, b]$. Prove that f is a measurable function.

[Hint: Let $f_n = f \cdot \chi_{[-n,n]}$. Show that $f_n \in L^1$ for all $n \geq 1$. Then show that $f_n \to f$ on all of $\mathbb{R}$.]