Homework #3
Solutions

Measure Zero!

① (a) Let $A, B \subseteq \mathbb{R}$ where $A \subseteq B$.

Suppose B has measure zero.
We will show that A has measure zero

Let $\varepsilon > 0$.
Since B has measure zero there exists a sequence of bounded open intervals $I_1, I_2, I_3, \ldots$ where

$$B \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} \ell(I_n) \leq \varepsilon$$

Since $A \subseteq B$ we have

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} \ell(I_n) \leq \varepsilon.$$

Thus, A has measure zero.

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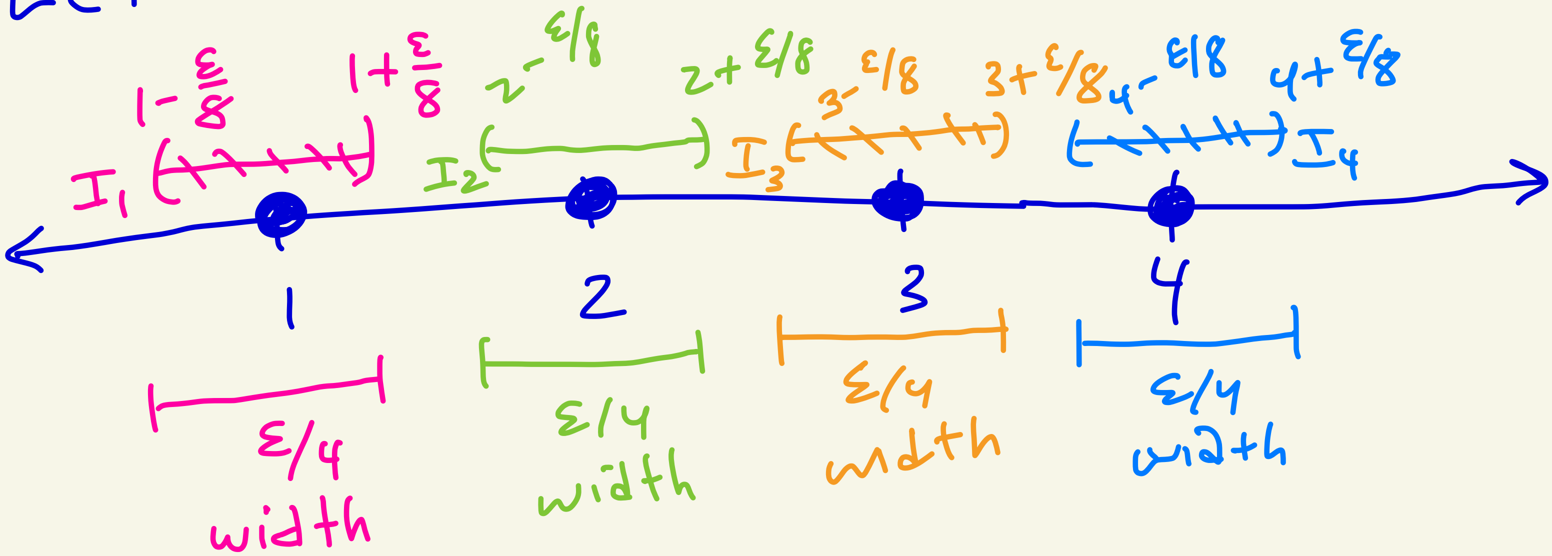
① (b) This is the converse of 1(a).
"If P, then Q" is equivalent to "If not Q, then not P"

② (a) S has measure zero.

Let $\varepsilon > 0$



Let
$$I_1 = \left(1-\frac{\varepsilon}{8}, 1+\frac{\varepsilon}{8}\right)$$
$$I_2 = \left(2-\frac{\varepsilon}{8}, 2+\frac{\varepsilon}{8}\right)$$
$$I_3 = \left(3-\frac{\varepsilon}{8}, 3+\frac{\varepsilon}{8}\right)$$
$$I_4 = \left(4-\frac{\varepsilon}{8}, 4+\frac{\varepsilon}{8}\right)$$

Then, $1 \in I_1$, $2 \in I_2$, $3 \in I_3$, $4 \in I_4$.

So, $S \subseteq \bigcup_{n=1}^{4} I_n$

And,

$$\sum_{n=1}^{4} \ell(I_n) = \ell(I_1) + \ell(I_2) + \ell(I_3) + \ell(I_4)$$

$$= \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

for ex, $\ell(I_1) = \left(1 - \frac{\varepsilon}{8}\right) - \left(1 + \frac{\varepsilon}{8}\right) = 2 \cdot \frac{\varepsilon}{8} = \frac{\varepsilon}{4}$

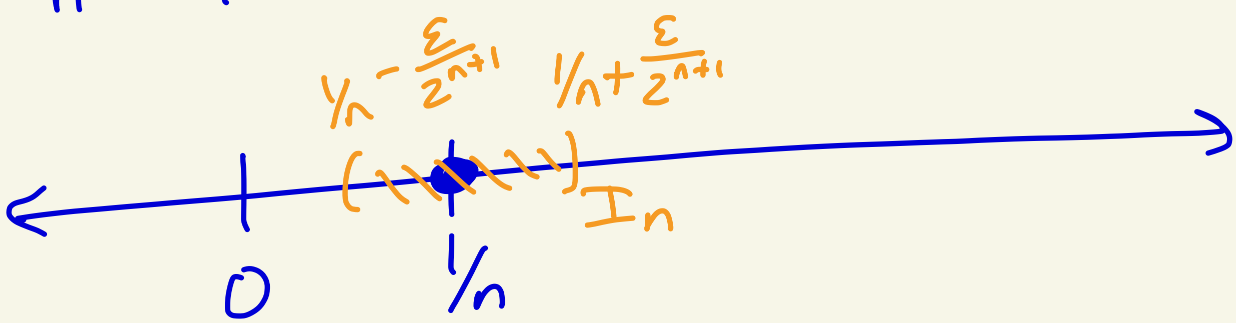
Thus, S has measure zero.

(2) (b) $S = \{ \frac{1}{n} \mid n = 1, 2, 3, 4, \ldots \}$

$= \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}$

For each $n \geq 1$, define

$$I_n = \left(\frac{1}{n} - \frac{\varepsilon}{2^{n+1}}, \frac{1}{n} + \frac{\varepsilon}{2^{n+1}} \right)$$



Then, $\frac{1}{n} \in I_n$ for $n \geq 1$.

Thus, $S \subseteq \bigcup_{n=1}^{\infty} I_n$.

Note that $\ell(I_n) = \left(\frac{1}{n} + \frac{\varepsilon}{2^{n+1}} \right) - \left(\frac{1}{n} - \frac{\varepsilon}{2^{n+1}} \right)$

$= 2 \cdot \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2^n}$

So,

$$\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \varepsilon\left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right]$$

$$= \frac{\varepsilon}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right]$$

$$= \frac{\varepsilon}{2}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{\varepsilon}{2} \cdot 2 = \varepsilon.$$

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

for $-1 < x < 1$

Thus, S has measure zero.

③ (a) $S = \{1, \pi, 10\}$ is finite, so S has measure zero.

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③ (b) $S = \mathbb{Q} \cap [0,5)$

Since $S \subseteq \mathbb{Q}$ and $\mathbb{Q}$ is countable, we know that S is countable.

Hence S has measure zero.

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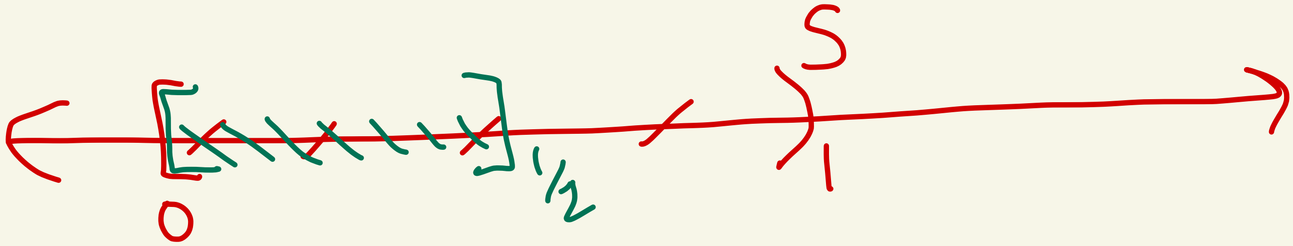
③ (c) We know from class that a set of the form $[a,b]$ where $a<b$ does not have measure zero.

Thus, $[0, 1/2]$ does not have measure zero.

From class we know that

We have $[0, \frac{1}{2}] \subseteq \underbrace{[0,1)}_{S}$

Thus,
by problem 1(b) of this HW assignment, we know $[0,1)$ does not have measure ~~zero~~.



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③(d) Let $\varepsilon = \frac{b-a}{4}$.

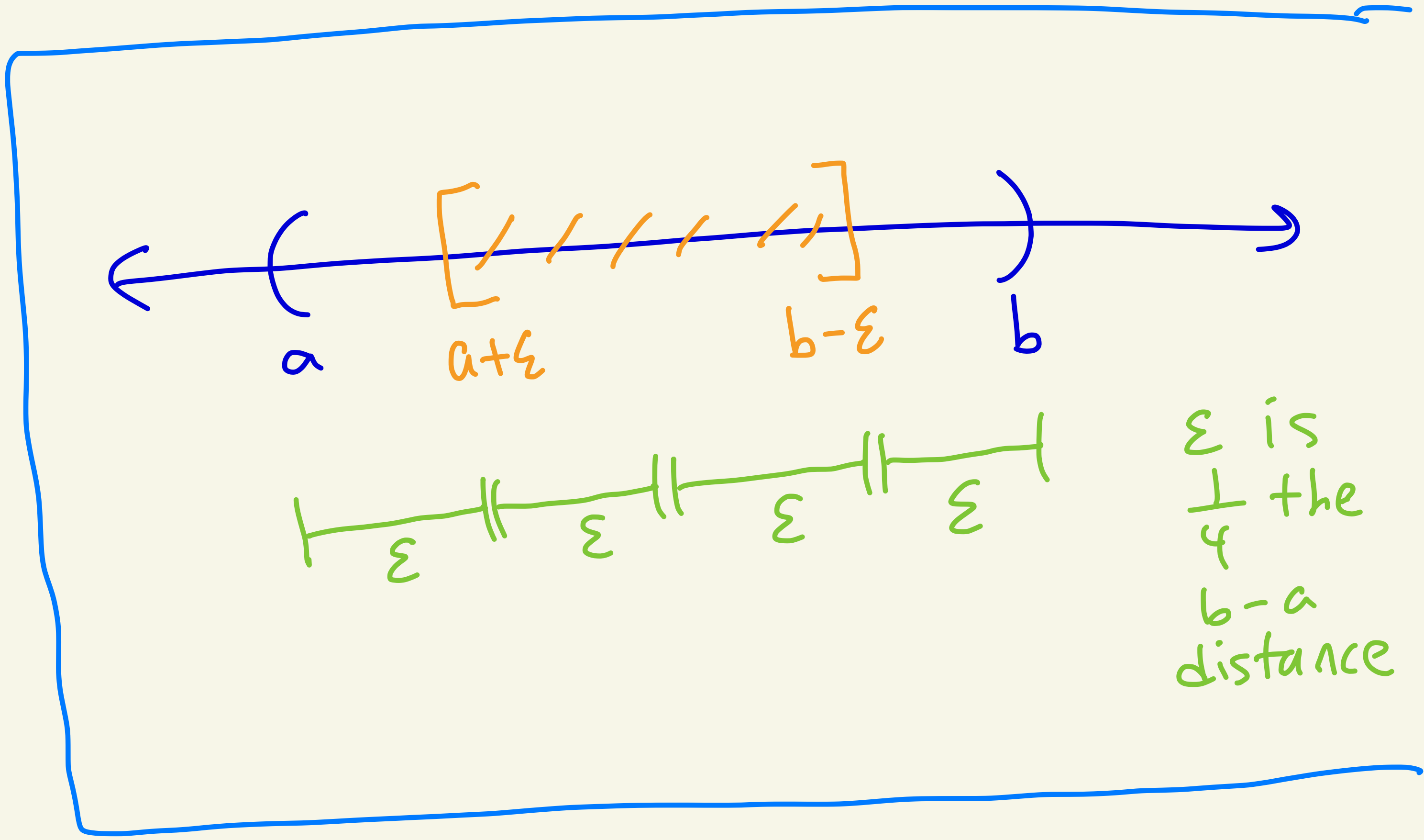
Note that

$$(b-\varepsilon)-(a+\varepsilon) = \left(b-\frac{b-a}{4}\right)-\left(a+\frac{b-a}{4}\right)$$
$$= b-\frac{1}{4}b+\frac{a}{4}-a-\frac{1}{4}b+\frac{a}{4}$$
$$= \frac{1}{2}b-\frac{1}{2}a > 0$$

since $b > a$

Thus, $a+\varepsilon < b-\varepsilon$ and so $[a+\varepsilon, b-\varepsilon]$ is a well-defined interval.

Note that $[a+\varepsilon, b-\varepsilon] \subseteq (a,b)$



Since $[a+\varepsilon, b-\varepsilon]$ does not have measure zero (by class) and $[a+\varepsilon, b-\varepsilon] \subseteq (a,b)$ we know from problem 1(b) of this HW that (a,b) does not have measure zero.

④ (a)

Since $S_1, S_2, \ldots, S_n$ are almost everywhere sets, we know that

$\mathbb{R}-S_1, \mathbb{R}-S_2, \ldots, \mathbb{R}-S_n$

have measure zero.

Thus, from a theorem in class, we know that $\bigcup_{k=1}^{n}(\mathbb{R}-S_k)$

has measure zero.

DeMorgan's law [Math 3450] tells us that

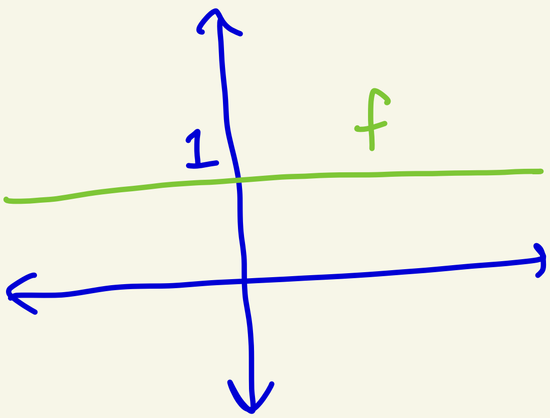
$$\mathbb{R}-\bigcap_{k=1}^{n} S_k = \bigcup_{k=1}^{n}(\mathbb{R}-S_k)$$

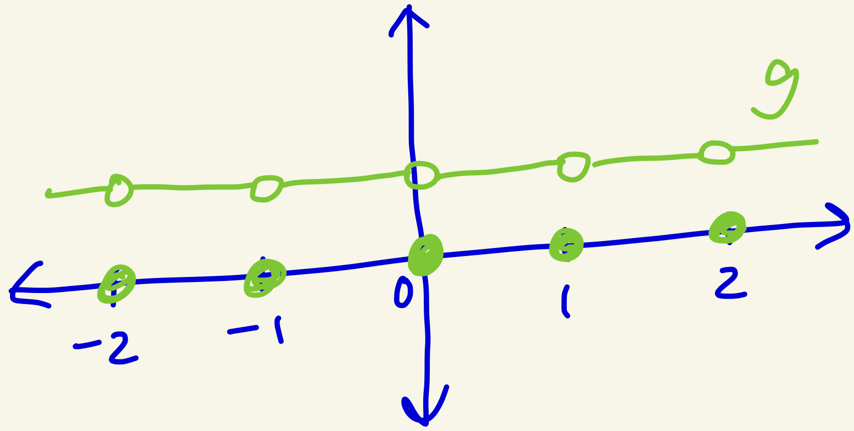
Thus, $\mathbb{R}-\bigcap_{k=1}^{n} S_k$ has measure zero.

Thus, $\bigcap_{k=1}^{n} S_k$ is an almost everywhere set.

4 (b) Same proof as 4(a) but turn $\bigcap_{k=1}^{n} S_k$ into $\bigcap_{k=1}^{\infty} S_k$

5 (a)



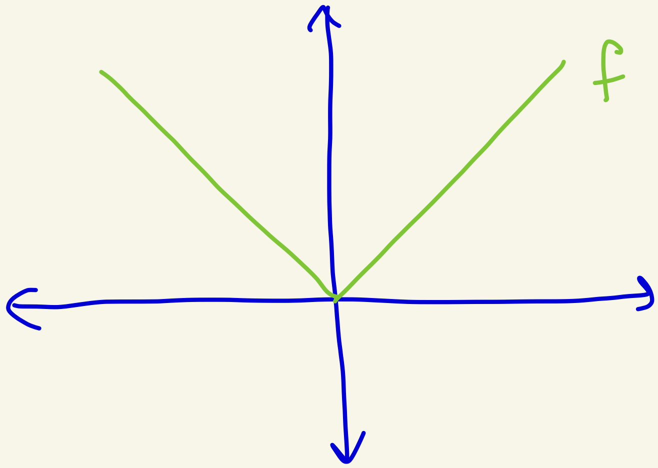


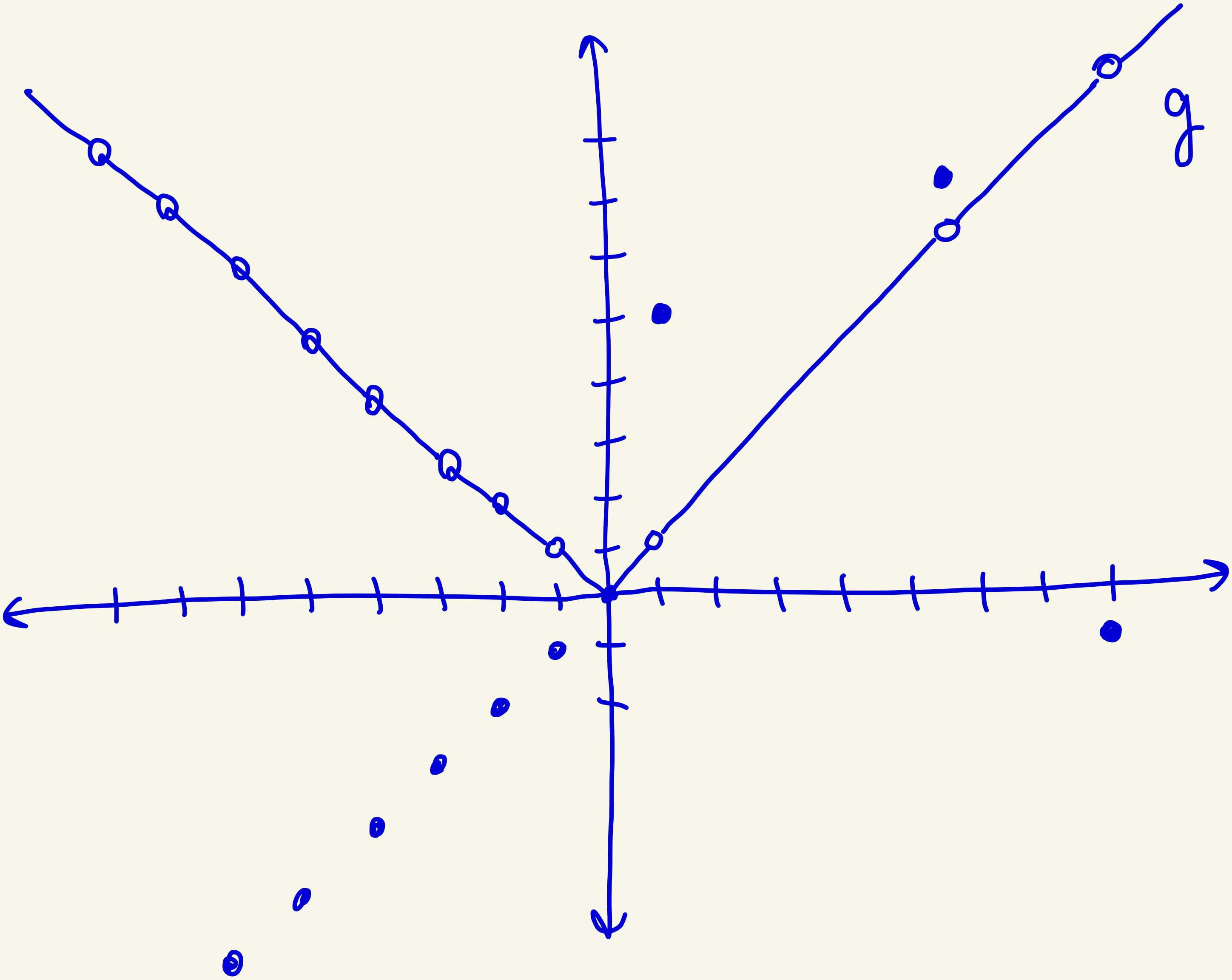
And $f(x) \neq g(x)$ iff $x \in \mathbb{Z}$.

$\mathbb{Z}$ is countable and hence has measure zero.

Since $f(x) = g(x)$ except on the set $\mathbb{Z}$ of measure zero we have that $f = g$ almost everywhere.

⑤ (b)

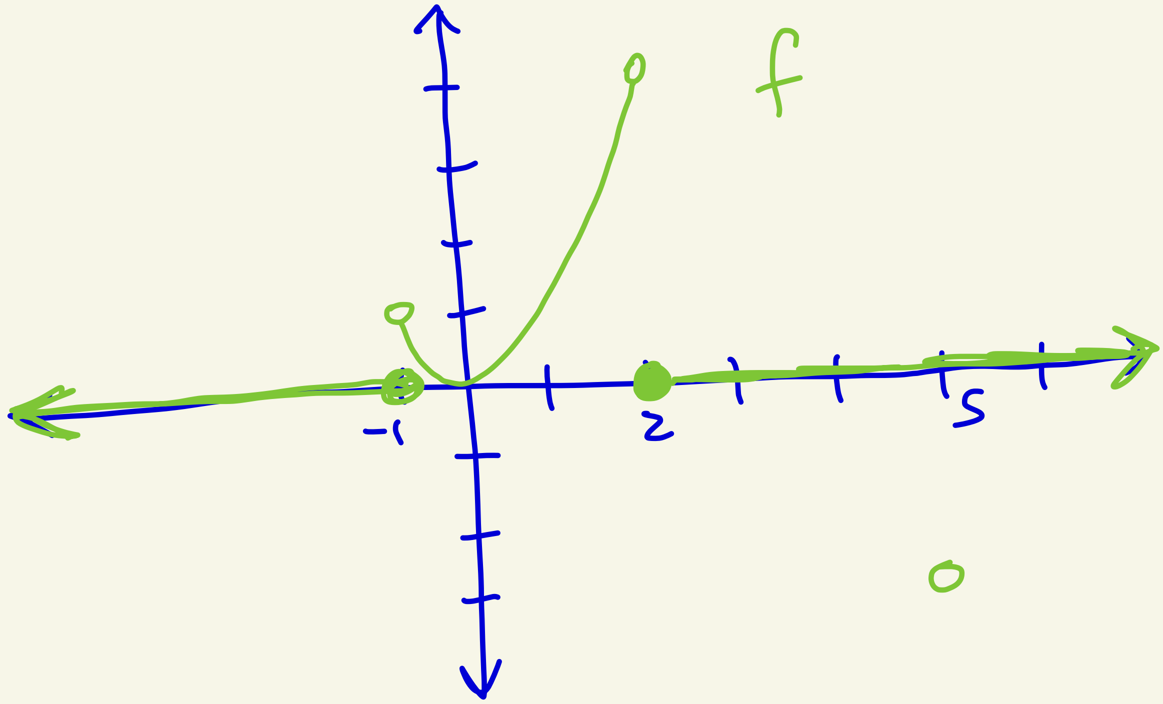


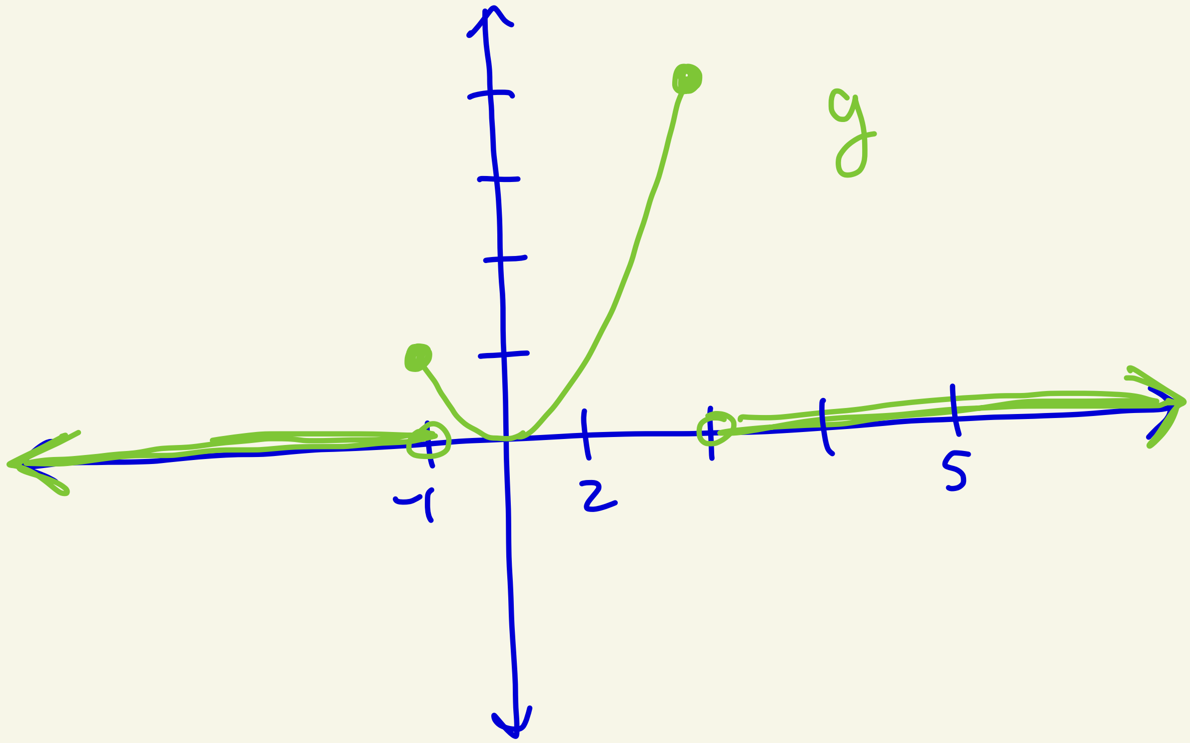


Note that $f(x) \neq g(x)$ iff $x \in \{1, 6, 8\} \cup \{-1, -2, -3, \ldots\}$

So, $f(x) \neq g(x)$ on a countable and hence measure zero set. Thus $f = g$ almost everywhere.

5 (c)





$f(x) \neq g(x)$ iff $x = -1, 2, 5$.

So, $f(x) = g(x)$ except on a countable and hence measure zero set.

So, $f = g$ almost everywhere.

5 (d) $f(x) \neq g(x)$ iff $x \notin \mathbb{Z}$
iff $x \in \mathbb{R} - \mathbb{Z}$

$\mathbb{R} - \mathbb{Z}$ does not have measure zero since for example $(0,1) \subseteq \mathbb{R} - \mathbb{Z}$ and $(0,1)$ does not have measure zero from problem 3(d).

Thus, $f \neq g$ almost everywhere.

⑥ (a) Let $A \subseteq B$ where A is an almost everywhere set.

Then $\mathbb{R} - A$ has measure zero.

But $\mathbb{R} - B \subseteq \mathbb{R} - A$, and hence

since $A \subseteq B$

by problem 1(a) of this HW, we know $\mathbb{R} - B$ has measure zero.

So, B is an almost everywhere set.

⑥ (b)

Since $f=g$ almost everywhere in $\mathbb{R}$, we know that

$$E_1 = \{x \mid f(x) = g(x)\}$$

is an almost everywhere set, ie $\mathbb{R}-E_1$ has measure zero.

Since $h(x)=5$ almost everywhere in $\mathbb{R}$, we know that

$$E_2 = \{x \mid h(x) = 5\}$$

is an almost everywhere set, ie $\mathbb{R}-E_2$ has measure zero.

Claim: $E_1 \cap E_2$ is an almost everywhere set.

pf of claim: $\mathbb{R} - (E_1 \cap E_2)$

$= (\mathbb{R} - E_1) \cup (\mathbb{R} - E_2)$

has measure zero since $\mathbb{R} - E_1$ and $\mathbb{R} - E_2$ have measure zero and hence their union has measure zero.

Thus $E_1 \cap E_2$ is an almost everywhere set. $\boxed{\text{claim}}$

Thus,

$E_1 \cap E_2 = \{x \mid f(x) = g(x) \text{ and } h(x) = 5\}$

is an almost everywhere set and if $x \in E_1 \cap E_2$ then $f(x) = g(x)$ and $h(x) = 5$ and thus $f(x) + h(x) = g(x) + 5$.

Now,

$$B = \{x \mid f(x) + h(x) = g(x) + 5\}$$

satisfies $E_1 \cap E_2 \subseteq B$.

Since $E_1 \cap E_2$ is an almost everywhere set, by part (a), B is an almost everywhere set.

Thus, $f(x) + h(x) = g(x) + 5$ for almost all x.