Homework #5 Solutions

More 4650 Review

(1)

($\Rightarrow$) Let b be the infimum of S.

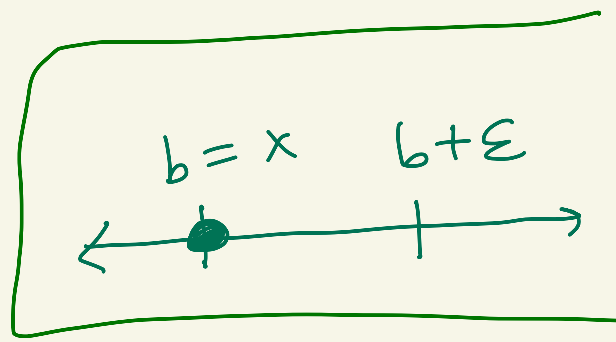
Then by def, b is a lower bound for S.

So, (i) is true.

Now lets show (ii).

Let $\varepsilon > 0$.

Case 1: Suppose $b \in S$.



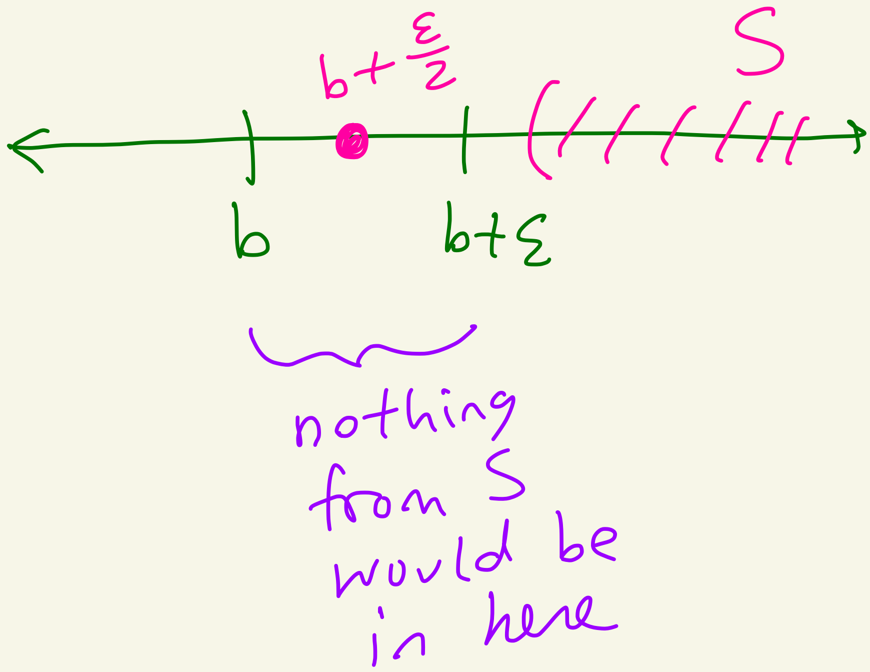
Set $x = b$.

Then $x \in S$ and $b \leq x < b+\varepsilon$ is satisfied.

Case 2: Suppose $b \notin S$.

What would happen if there was no $x \in S$ satisfying $b \leq x < b+\varepsilon$?

Then b would no longer be a lower bound for S.



Why?

Set $x = b + \frac{\varepsilon}{2}$.

Then $b \leq x < b+\varepsilon$.

And x would be a lower bound for S, because no elements of S are in the interval $\left[b, b+\frac{\varepsilon}{2}\right]$.

This would contradict b being the infimum of S because x would be a greater lower bound than b.

Thus, for case (iii), we must have that there exists $x \in S$ with $b \geq x > b+\varepsilon$.

Therefore we have show (i) and (ii).
And the proof is complete.

($\Leftarrow$) Suppose $b \in \mathbb{R}$ and

(i) b is a lower bound for S, and

(ii) for every $\varepsilon > 0$ there exists $x \in S$ with $b \le x < b + \varepsilon$.

are true.

We must show that b is the infimum of S.

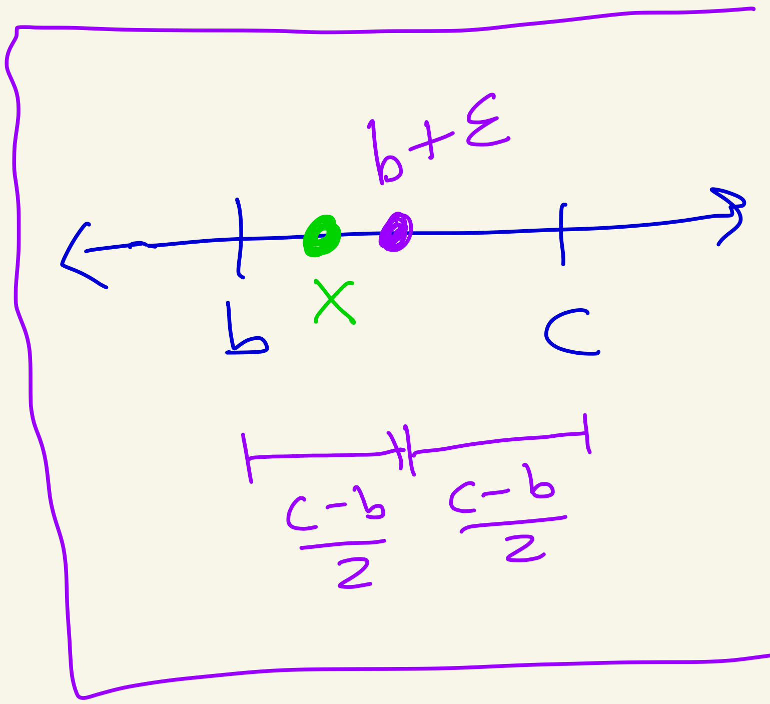
We know b is a lower bound for S.

We must show that it is the greatest lower bound for S.

Let c be another lower bound for S.

We must show that $c \leq b$.
Suppose otherwise, that is suppose that $b < c$.

Let $\varepsilon = \frac{c-b}{2}$.



Because ε is half the distance between c & b we have that $b < b+\varepsilon < c$.

By property (ii) there would then exist $x \in S$ with $b \leq x < b+\varepsilon$.

But then $x \in S$ and $x < b+\varepsilon < c$.

This would contradict the fact that c is a lower bound for S.

Thus, we must have $c \leq b$.

We have show that b is the infimum of S. ▨

② Let $(a_n)_{n=1}^{\infty}$ be a non-decreasing sequence where $a_n \leq M$ for all $n \geq 1$ for some $M \in \mathbb{R}$.

Let $S = \{ a_k \mid k = 1, 2, 3, \ldots \}$
$= \{ a_1, a_2, a_3, a_4, \ldots \}$

Then M is an upper bound for S.
Thus, by the completeness axiom for $\mathbb{R}$ we know that the supremum of S exists.

Let $L = \sup(S)$.
We will show that $\lim_{n \to \infty} a_n = L$.

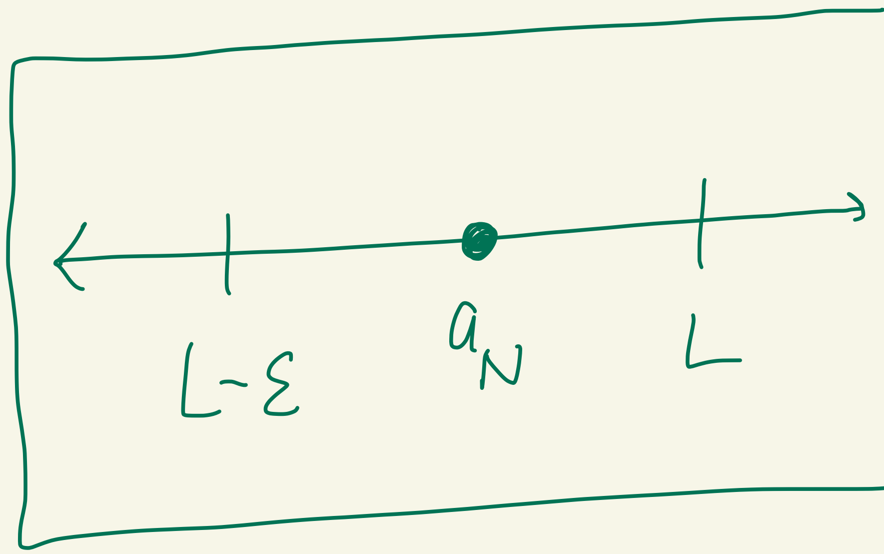
Let $\varepsilon > 0$.

Since L is the supremum of S, there exists $a_N \in S$ where

$$L - \varepsilon < a_N \leq L.$$

Suppose $n \geq N$.

Because $(a_n)_{n=1}^{\infty}$ is non-decreasing we know



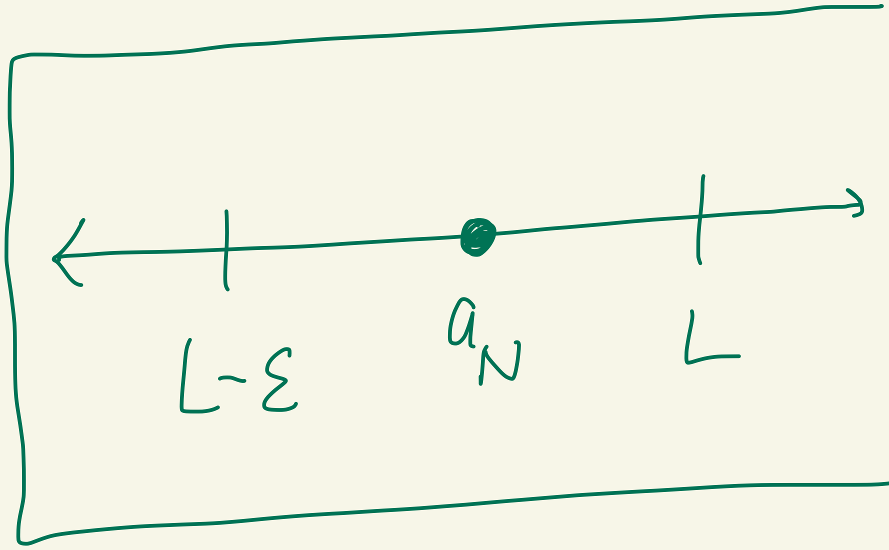
$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_N \leq a_{N+1} \leq \ldots \leq a_n \leq \ldots$$

That is, since $n \geq N$ we have $a_N \leq a_n$.

Since $a_n \in S$, and L is an upper bound for S we know that $a_n \leq L$.

Summaring, if $n \geq N$ then

$$L - \varepsilon < a_N \leq a_n \leq L$$



So, if $n \geq N$, then

$$L - \varepsilon < a_n \leq L < L + \varepsilon.$$

That is, if $n \geq N$, then

$$\boxed{L - \varepsilon < a_n < L + \varepsilon,}$$

same as $|a_n - L| < \varepsilon$

So, $\lim\limits_{n \to \infty} a_n = L$ ▨