4680 - HW 8

Solutions

① (a)

Let $\varepsilon > 0$.

$$\frac{n-1}{n} - \frac{n}{n} = \frac{-1}{n}$$

$$1 - \frac{1}{n} - 1$$

Note that

$$|z_n - i| = \left| \frac{1}{n} + i\frac{n-1}{n} - i \right|$$

$$= \left| \frac{1}{n} - \frac{1}{n} i \right|$$

$$\leq \left| \frac{1}{n} \right| + \left| -\frac{1}{n} i \right|$$

$$= \left| \frac{1}{n} \right| + \left| -\frac{1}{n} \right| \underbrace{|i|}_{1}$$

$$= \frac{1}{n} + \frac{1}{n}$$

$$= \frac{2}{n}.$$

Note that $\frac{2}{n} < \varepsilon$ iff $\frac{2}{\varepsilon} < n$.

Let $N > \frac{2}{\varepsilon}$.

Then if $n \geq N > \frac{2}{\varepsilon}$ we have

that $|z_n - i| = \frac{2}{n} < \varepsilon$.

So, $\lim_{n\to\infty} z_n = i$.

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① (b) $z_n = \frac{1}{n} + i\left[\frac{n-1}{n}\right]$.

Let $x_n = \frac{1}{n}$ and $y_n = \frac{n-1}{n}$.

Then $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{n-1}{n}$

$= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1 - 0$

$= 1$.

By a thm in class,

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + i(1) = i$

$\boxed{2(a)}$

Let $\varepsilon > 0$.

Note that

$$|z_n - (-2)| = \left|\left(-2 + i\frac{(-1)^n}{n^2}\right) - (-2)\right|$$

$$= \left|i\frac{(-1)^n}{n^2}\right| = |i|\,\frac{|(-1)^n|}{|n^2|}$$

$$= \frac{1}{n^2}$$

Note that $\frac{1}{n^2} < \varepsilon$ iff $\frac{1}{\varepsilon} < n^2$

iff $\frac{1}{\sqrt{\varepsilon}} < n$.

Pick some $N > \frac{1}{\sqrt{\varepsilon}}$.

If $n \geq N > \frac{1}{\sqrt{\varepsilon}}$, then

$$|z_n - (-2)| = \frac{1}{n^2} < \varepsilon.$$

So, $z_n \to -2$.

$\boxed{2(b)}$

Note that
$$-2 \to -2$$

and
$$\frac{(-1)^n}{n^2} \to 0$$

Thus, by the thm in class,
$$z_n = -2 + i\frac{(-1)^n}{n^2}$$
$$\to -2 + i\,0 = -2.$$

③ Let $z_n = x_n + iy_n$ where $x_n, y_n \in \mathbb{R}$ for all n.

($\Longrightarrow$) Suppose that (z_n) is a Cauchy sequence. We will show that (x_n) and (y_n) are Cauchy sequences.

Let $\varepsilon > 0$. Since (z_n) is a Cauchy sequence, there exists $N > 0$ so that if $n, m \geq N$ then $|z_n - z_m| < \varepsilon$.

Note that $z_n - z_m = \underbrace{(x_n - x_m)}_{\mathrm{Re}(z_n - z_m)} + i\underbrace{(y_n - y_m)}_{\mathrm{Im}(z_n - z_m)}$

Thus, if $n, m \geq N$ then

$$|x_n - x_m| \leq |z_n - z_m| < \varepsilon.$$

$|\mathrm{Re}(w)| \leq |w|$

$w = z_n - z_m = (x_n - x_m) + i(y_n - y_m)$

Similarly if $n, m \geq N$ then

$$|y_n - y_m| \leq |z_n - z_m| < \varepsilon$$

$|Im(w)| \leq |w|$

$w = z_n - z_m = (x_n - x_m) + i(y_n - y_m)$

$(\Longleftarrow)$ Suppose that (x_n) and (y_n) are Cauchy sequences.

Let $\varepsilon > 0$.

Since (x_n) is Cauchy, there exists $N_1 > 0$ so that if $n, m \geq N_1$ then $|x_n - x_m| < \frac{\varepsilon}{2}$.

Since (y_n) is Cauchy, there exists $N_2 > 0$ so that if $n, m \geq N_2$ then $|y_n - y_m| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$.

If $n, m \geq N$, then

$$|z_n - z_m| = |(x_n - x_m) + i(y_n - y_m)|$$
$$\leq |x_n - x_m| + |i(y_n - y_m)|$$
$$= |x_n - x_m| + |i||y_n - y_m|$$
$$= |x_n - x_m| + |y_n - y_m|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

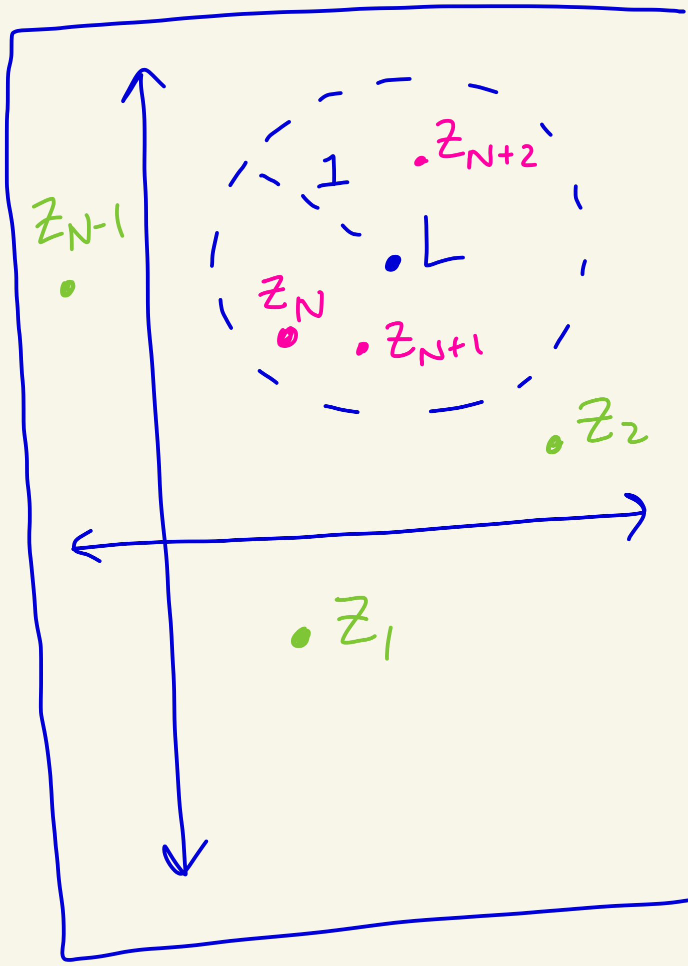
$\blacksquare$

④ Suppose $(z_n)_{n=1}^{\infty}$ converges L.

Let $\varepsilon = 1$.

Then, there exists an integer $N > 0$ so that if $n \geq N$ we have that

$$|z_n - L| < 1.$$



Thus if $n \geq N$, then

$$|z_n| = |z_n - L + L|$$
$$\leq |z_n - L| + |L|$$
$$< 1 + |L|.$$

Let

$$M = \max\{|z_1|, |z_2|, \ldots, |z_{N-1}|, 1 + |L|\}.$$

Consider z_n for some n.

If $1 \leq n \leq N-1$, then $|z_n| \leq M$.

If $n \geq N$, then $|z_n| \leq 1 + |L| \leq M$.

Hence, $|z_n| \leq M$ for all n.

Method 1 for #5

For #4, we use this fact from class:

Suppose $z_n = x_n + iy_n$ and $L = x_0 + iy_0$.

$z_n \to L$ iff both $\underbrace{x_n \to x_0 \text{ and } y_n \to y_0}_{\text{real analysis limits}}$

(5) Suppose $z_n = x_n + iy_n$ and

$w_n = a_n + ib_n$ and $A = x_0 + iy_0$

and $B = a_0 + ib_0$

Suppose $z_n \to A$ and $w_n \to B$.

Then, $x_n \to x_0$, $y_n \to y_0$, $a_n \to a_0$, and $b_n \to b_0$.

(a) Let $\alpha = \alpha_1 + i\alpha_2$ and

$\beta = \beta_1 + i\beta_2$.

Note that

$$\alpha z_n + \beta w_n = (\alpha_1 + i\alpha_2)(x_n + iy_n) + (\beta_1 + i\beta_2)(a_n + ib_n)$$

$$= \alpha_1 x_n - \alpha_2 y_n + i(\alpha_2 x_n + \alpha_1 y_n) + \beta_1 a_n - \beta_2 b_n + i(\beta_2 a_n + \beta_1 b_n)$$

$$= (\alpha_1 x_n - \alpha_2 y_n + \beta_1 a_n - \beta_2 b_n) + i(\alpha_2 x_n + \alpha_1 y_n + \beta_2 a_n + \beta_1 b_n)$$

Since $x_n \to x_0$, $y_n \to y_0$, $a_n \to a_0$, $b_n \to b_0$ we have that

$$\alpha_1 x_n - \alpha_2 y_n + \beta_1 a_n - \beta_2 b_n \to \alpha_1 x_0 - \alpha_2 y_0 + \beta_1 a_0 - \beta_2 b_0$$

and

$$\alpha_2 x_n + \alpha_1 y_n + \beta_2 a_n + \beta_1 b_n \to \alpha_2 x_0 + \alpha_1 y_0 + \beta_2 a_0 + \beta_1 b_0$$

Therefore, by the thm in class (and before this solution) we have that $\alpha z_n + \beta w_n$ converges to

$$(\alpha_1 x_0 - \alpha_2 y_0 + \beta_1 a_0 - \beta_2 b_0)$$
$$+ i(\alpha_2 x_0 + \alpha_1 y_0 + \beta_2 a_0 + \beta_1 b_0)$$

$$= (\alpha_1 + i\alpha_2)(x_0 + i y_0)$$
$$+ (\beta_1 + i\beta_2)(a_0 + i b_0)$$

$$= \alpha A + \beta B.$$

(b) Note that

$$z_n w_n = (x_n + i y_n)(a_n + i b_n)$$
$$= (x_n a_n - y_n b_n) + i(x_n b_n + y_n a_n)$$

Since $x_n \to x_0$, $y_n \to y_0$, $a_n \to a_0$, $b_n \to b_0$ we have that

$$x_n a_n - y_n b_n \to x_0 a_0 - y_0 b_0$$

and

$$x_n b_n + y_n a_n \to x_0 b_0 + y_0 a_0$$

By the thm in class (or before this solution) we have that

$$z_n w_n = (x_n a_n - y_n b_n) + i(x_n b_n + y_n a_n)$$

$$\to (x_0 a_0 - y_0 b_0) + i(x_0 b_0 + y_0 a_0)$$

$$= (x_0 + i y_0)(a_0 + i b_0)$$

$$= AB$$

Method #2 for problem 5

5(a) Let $\varepsilon > 0$.

Note that

$$|\alpha z_n + \beta w_n - (\alpha A + \beta B)|$$
$$= |(\alpha z_n - \alpha A) + (\beta w_n - \beta B)|$$
$$\leq |\alpha z_n - \alpha A| + |\beta w_n - \beta B|$$
$$= |\alpha||z_n - A| + |\beta||w_n - B|$$
$$= |\alpha||z_n - A| + (|\beta| + 1)|w_n - B|$$
$$< (|\alpha| + 1)|z_n - A| + (|\beta|+1)$$

[We are putting $|\alpha|+1$ and $|\beta|+1$ because we will divide by this number we want it to be non-zero and we could have $|\alpha| = 0$ or $|\beta| = 0$ so thats why we replace them by $|\alpha|+1 > 0$ and $|\beta|+1 > 0$.]

Since $\lim_{n\to\infty} z_n = A$ and $\lim_{n\to\infty} w_n = B$

there exists $N > 0$ such that

if $n \geq N$ then

$$|z_n - A| < \frac{\varepsilon}{2(|\alpha|+1)}$$

and

$$|w_n - B| < \frac{\varepsilon}{2(|\beta|+1)}.$$

Thus, if $n \geq N$ we have that

$$|\alpha z_n + \beta w_n - (\alpha A + \beta B)|$$

$$< (|\alpha|+1)|z_n - A| + (|\beta|+1)|w_n - B|$$

$$< (|\alpha|+1)\frac{\varepsilon}{2(|\alpha|+1)} + (|\beta|+1)\frac{\varepsilon}{2(|\beta|+1)}$$

$$= \varepsilon.$$ So, $\alpha z_n + \beta w_n \to \alpha A + \beta B$.

$\boxed{5(b)}$ Let $\varepsilon > 0$.

Note that

$$
\begin{aligned}
&|z_n w_n - AB| \\
&= |z_n w_n - A w_n + A w_n - AB| \\
&\leq |z_n w_n - A w_n| + |A w_n - AB| \\
&= |w_n||z_n - A| + |A||w_n - B| \\
&< |w_n||z_n - A| + \underbrace{(|A|+1)}_{\text{we want a non-zero number here}} |w_n - B|
\end{aligned}
$$

Since (w_n) converges, by the previous HW problem (w_n) is bounded so there exists $M > 0$ so that $|w_n| \leq M$ for all n.

Since $z_n \to A$ and $w_n \to B$ there exists $N > 0$ so that if $n \geq N$ we have that

$$|z_n - A| < \frac{\varepsilon}{2M}$$

and

$$|w_n - B| < \frac{\varepsilon}{2(|A|+1)}.$$

Thus, if $n \geq N$ then

$$|z_n w_n - AB|$$
$$< |w_n||z_n - A| + (|A|+1)|w_n - B|$$
$$< M \cdot \frac{\varepsilon}{2M} + (|A|+1)\frac{\varepsilon}{2(|A|+1)}$$
$$= \varepsilon.$$ So, $z_n w_n \to AB$.

⑥ ($\Rightarrow$) Suppose F is closed.

Then $\mathbb{C}-F$ is open.

Let $(z_n)_{n=1}^{\infty}$ be a sequence of points in F.

Suppose $w = \lim\limits_{n\to\infty} z_n$ exists.

Let's show that $w \in F$.

Suppose $w \notin F$.

Then $w \in \mathbb{C}-F$ which is open.

Then w would be an interior point of $\mathbb{C}-F$.

So there would exist $r > 0$ such that
$D(w;r) \subseteq \mathbb{C}-F$.

But since $\lim\limits_{n\to\infty} z_n = w$, there exists $N > 0$ such that if $n \geq N$ then $\underbrace{|z_n - w| < r}_{\text{This means } z_n \in D(w;r)}$.

Thus there exists some $z_n \in D(w;r)$.

This contradicts the fact ↓

that $D(w;r) \subseteq \mathbb{C} - F$
since $z_n \in F$.

Thus, we must have that w is in fact in F.

$(\Leftarrow)$ Suppose that whenever a sequence of points $(z_n)_{n=1}^{\infty}$ in F converges and $w = \lim_{n \to \infty} z_n$, then $w \in F$.

Let's show F is closed.

We show that F not being closed leads to a contradiction.

Suppose F is not closed.
Then $\mathbb{C} - F$ is not open.
So there exists $w \in \mathbb{C} - F$ where w is not an interior point of $\mathbb{C} - F$.

Thus, for every $n \geq 1$, $D(w; \frac{1}{n}) \not\subseteq \mathbb{C} - F$.

So, for every $n \geq 1$,
we can find $z_n \in D(w; \frac{1}{n})$
such that $z_n \notin \mathbb{C} - F$,
ie $z_n \in F$.

Thus, we can construct
a sequence of points
$(z_n)_{n=1}^{\infty}$ from F such that
$\underbrace{|z_n - w| \leq \frac{1}{n}}_{z_n \in D(w; \frac{1}{n})}$.

I claim then that $\lim_{n \to \infty} z_n = w$.

Let $\varepsilon > 0$.
Pick $N > 0$ such that $\frac{1}{N} < \varepsilon$.
Then if $n \geq N$ we have that
$\frac{1}{n} \leq \frac{1}{N}$ and so $|z_n - w| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

So, $\lim_{n \to \infty} z_n = w$ but $w \notin F$.

Contradiction.

Hence, F is closed.

⑦ We use this fact from HW.

Let $F \subseteq \mathbb{C}$. Then F is closed iff whenever $(z_n)_{n=1}^{\infty}$ is a sequence of points in F such that $\lim_{n\to\infty} z_n = w$ exists, then $w \in F$

Suppose $(z_n)_{n=1}^{\infty}$ is a sequence of points on $\gamma([a,b])$ such that $w = \lim_{n\to\infty} z_n$ exists.

We need to show that $w \in \gamma([a,b])$.

Define the sequence $(t_n)_{n=1}^{\infty}$ in $[a,b]$ where $\gamma(t_n) = z_n$ for each $n \geq 1$.

Since $a \leq t_n \leq b$, (t_n) is a bounded sequence in $\mathbb{R}$.

So by Bolzano-Weierstrass there exists a subsequence (t_{n_k}) that converges to some $\hat{t} \in \mathbb{R}$.

Since $[a,b]$ is a closed set in $\mathbb{R}$, we have that $\hat{t} \in [a,b]$.

Since (z_{n_k}) is a subsequence of (z_n), we have that $\lim z_{n_k} = w$.

<u>Claim:</u> $\lim\limits_{n_k \to \infty} \gamma(t_{n_k}) = \gamma(\hat{t})$

<u>pf:</u> Let $\varepsilon > 0$.

Since $\hat{t} \in [a,b]$ and γ is continuous on $[a,b]$, there exists $\delta > 0$ so that if $t \in [a,b]$, $|t - \hat{t}| < \delta$ then $|\gamma(t) - \gamma(\hat{t})| < \varepsilon$

Since $t_{n_k} \to \hat{t}$, there exists $N > 0$ so that if $n_k \geq N$ then $|t_{n_k} - \hat{t}| < \delta$.

Thus, if $n_k \geq N$ we have that $|t_{n_k} - \hat{t}| < \delta$ and so $|\gamma(t_{n_k}) - \gamma(\hat{t})| < \varepsilon$.

Thus, $\gamma(t_{n_k}) \to \gamma(\hat{t})$

$\boxed{\text{Claim}}$

Therefore,

$$w = \lim_{n_k \to \infty} z_{n_k} = \lim_{n_k \to \infty} \gamma(t_{n_k}) = \gamma(\hat{t}).$$

And $\gamma(\hat{t}) \in \gamma([a,b])$ since $\hat{t} \in [a,b]$.

So, $w \in \gamma([a,b])$.

Therefore, $\gamma([a,b])$ is closed.