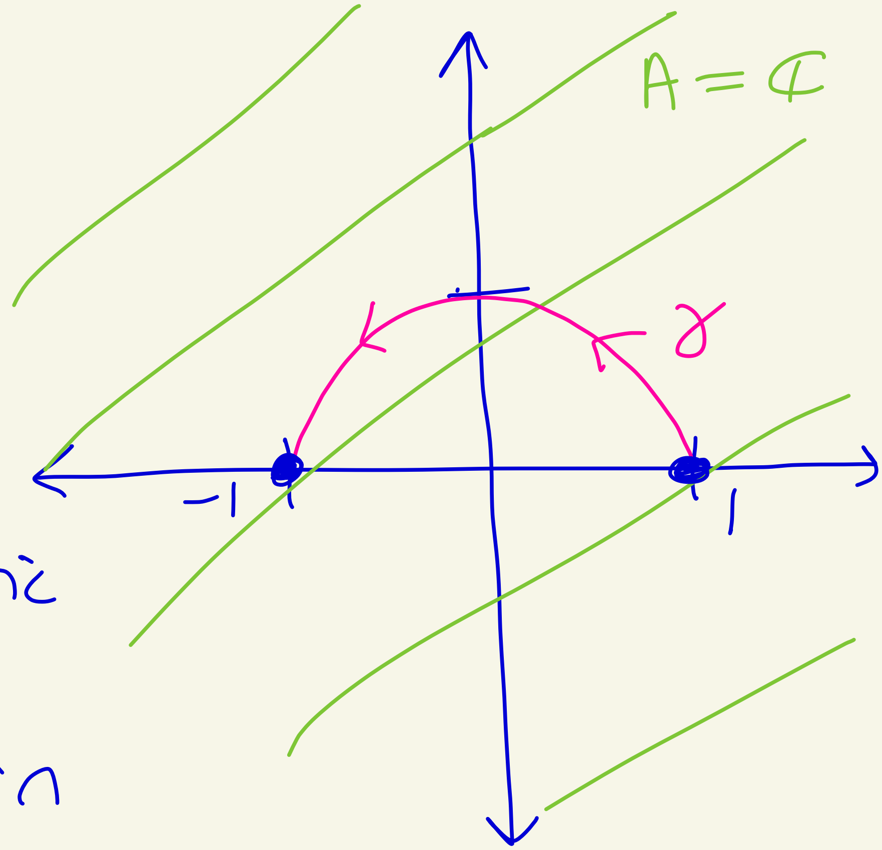
4680 - HW 9

Solutions

(1)(a)



Note that $f(z)=z^2-z+10$ is a polynomial so $f(z)$ is analytic on $A=\mathbb{C}$. Since γ lives in $\mathbb{C}$, we can use FTOC to get that

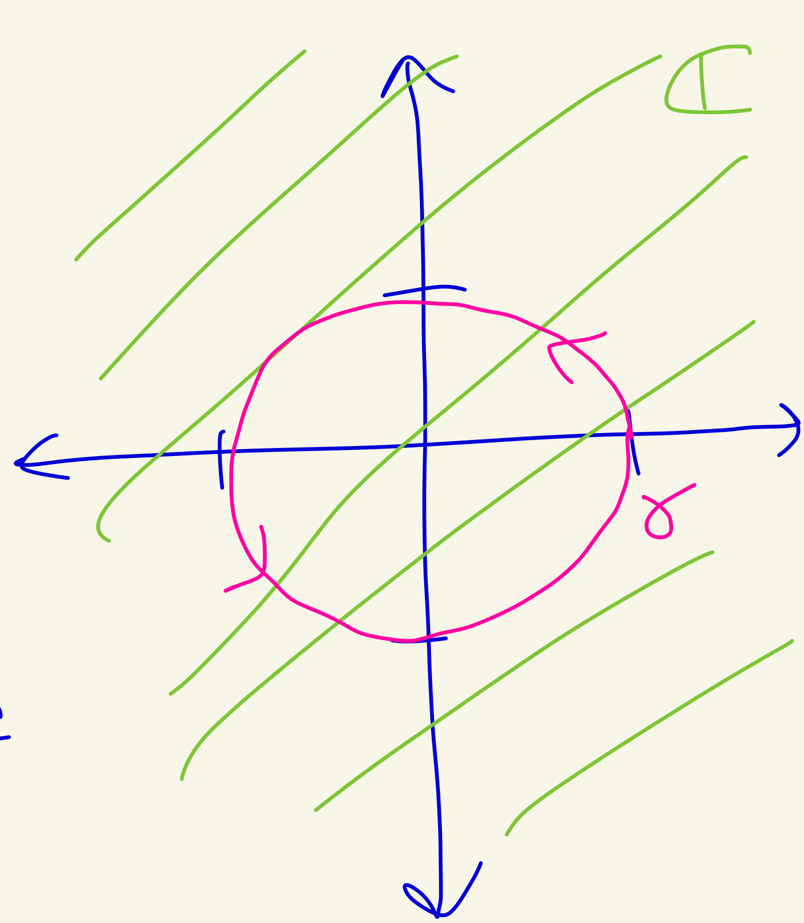
$$\int_\gamma (z^2-z+10)\,dz = \left(\frac{z^3}{3}-\frac{z^2}{2}+10z\right)\Bigg|_1^{-1}$$

$$= \left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}+10(-1)\right)-\left(\frac{1^3}{3}-\frac{1^2}{2}+10(1)\right)$$

$$= -\frac{1}{3}-\frac{1}{2}-10-\frac{1}{3}+\frac{1}{2}-10$$

$$= -\frac{2}{3}-20 = \frac{-2-60}{3} = \frac{-62}{3}$$

① (b) The unit circle γ is a simple, closed, smooth curve.



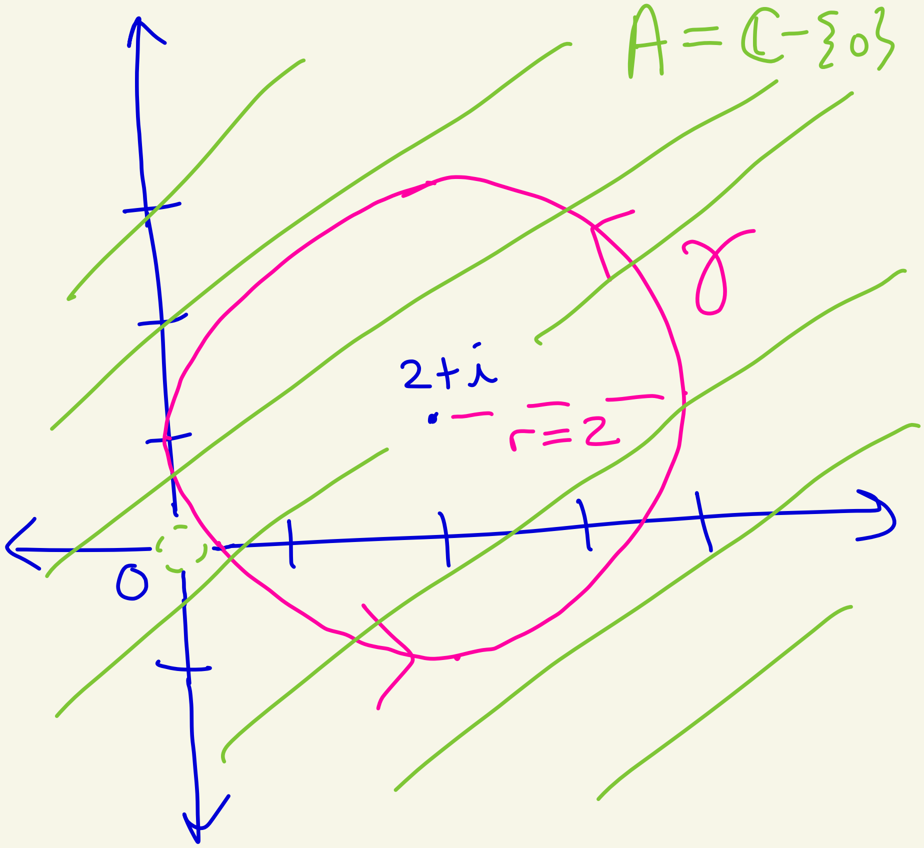
The function $f(z) = z^2 - z + 10$ is analytic on all of $\mathbb{C}$ and hence is analytic on the unit circle γ and inside γ.

So, by Cauchy's thm

$$\int_\gamma (z^2 - z + 10) = 0$$

[You could also use FTOC.]

1 (c)



The function $f(z) = e^{1/z}$ is analytic everywhere except at $z = 0$.

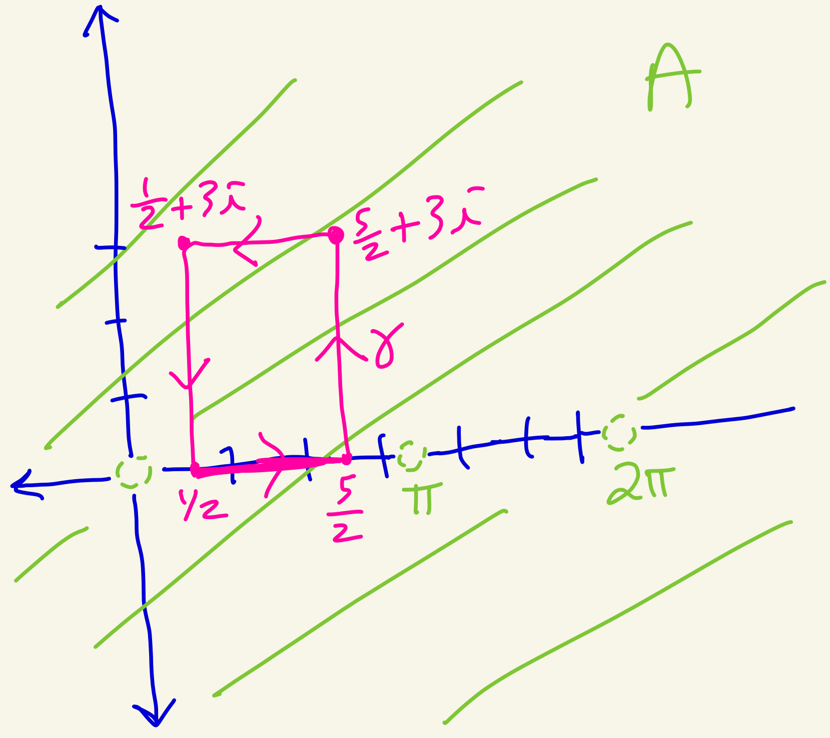
That is f is analytic on $A = \mathbb{C} - \{0\}$

Since f is analytic on and inside of γ, by Cauchy's theorem, $\int_\gamma e^{1/z}\, dz = 0$

1 (d)

From HW 2, $\sin(z)=0$ iff $z=\pi k$ for $k\in\mathbb{Z}$.

Let $A=\mathbb{C}-\{\pi k \mid k\in\mathbb{Z}\}$.



Then $f(z)=\dfrac{1}{\sin(z)}$ is analytic on A.

Since f is analytic on γ and inside γ, by Cauchy's theorem,

$$\int_\gamma \frac{1}{\sin(z)}\,dz = 0$$

①(e)

Define

$f(z) = z^i$

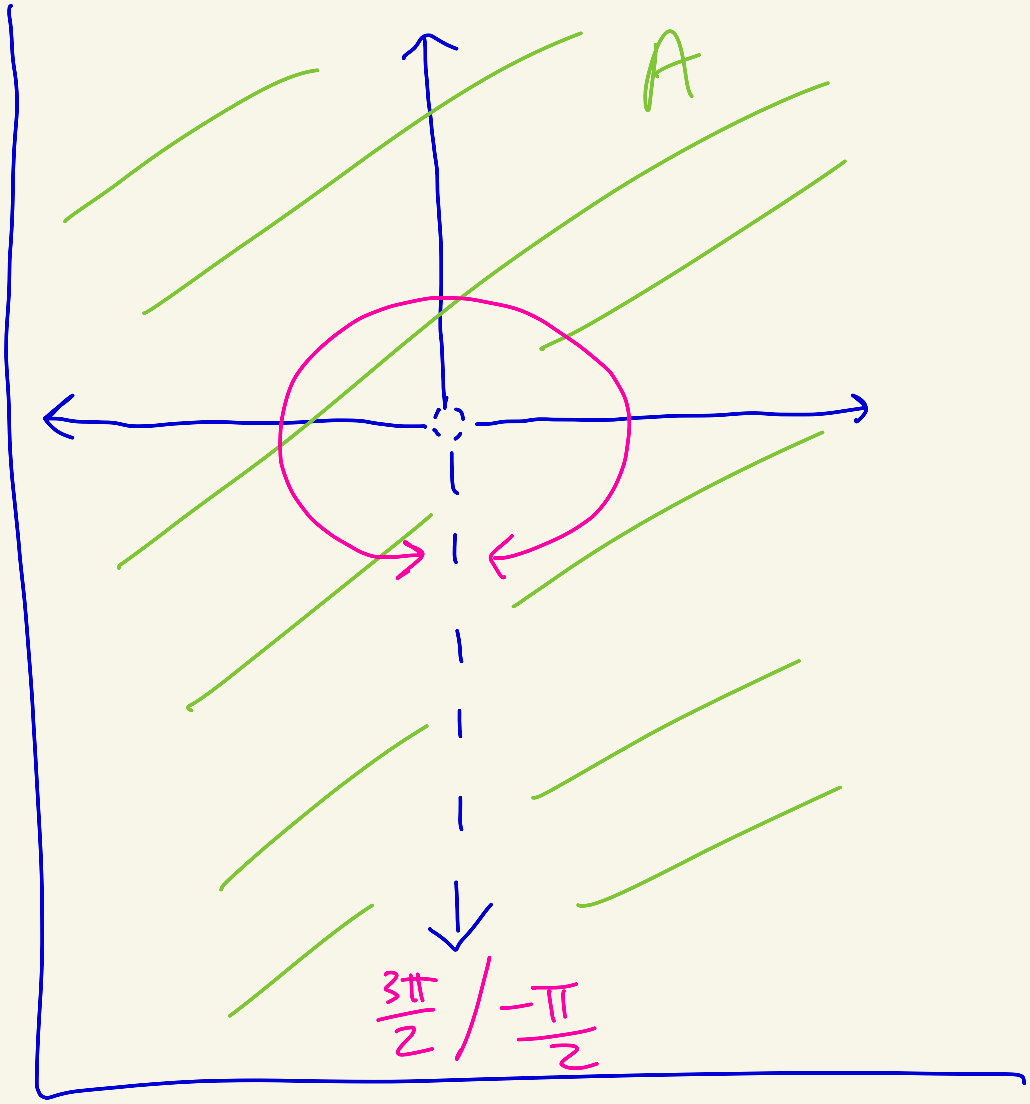
$= e^{i\log(z)}$

where

$\log(z) = \ln|z| + i\arg(z)$

and

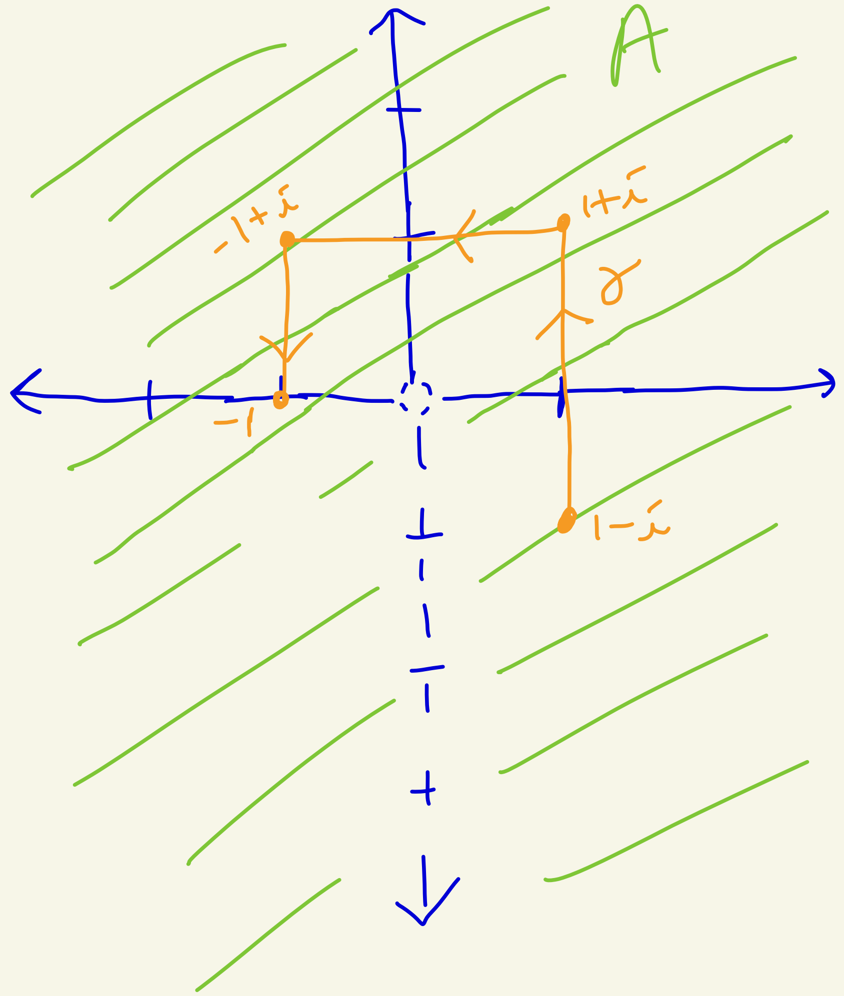
$-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$



From class, $f(z) = z^i$, with this branch of logarithm is analytic on $A = \mathbb{C} - \{x+iy \mid x=0 \text{ and } y \le 0\}$.

Note that A is an open set containing γ and $f(z) = z^i$ and is analytic on γ and its antiderivative

$$g(z) = \frac{z^{i+1}}{i+1}$$ is continuous on A,



Thus by FTOC,

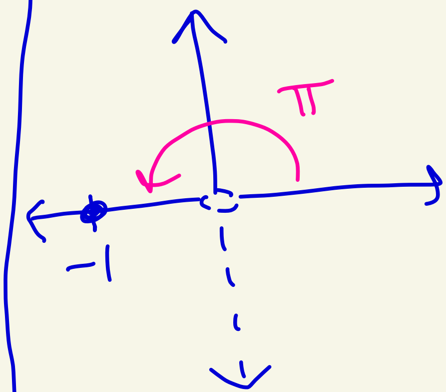
$$\int_\gamma z^i \, dz = \frac{z^{i+1}}{i+1} \Bigg|_{1-i}^{-1} = \frac{1}{i+1}(-1)^{i+1} - \frac{1}{i+1}(1-i)^{i+1}$$

$$= \frac{1}{i+1}\left[e^{(i+1)\log(-1)} - e^{(i+1)\log(1-i)}\right]$$

$$e^{(1+i)\log(-1)} = e^{(1+i)\left[\ln|-1| + i\pi\right]}$$

$$= e^{i\pi - \pi} = e^{-\pi}\left[\underbrace{\cos(\pi)}_{-1} + i\underbrace{\sin(\pi)}_{0}\right]$$

$$= -e^{-\pi}$$



$$\arg(-1) = \pi$$

$$-\frac{\pi}{2} < \pi < \frac{3\pi}{2}$$

$$|-1| = 1$$

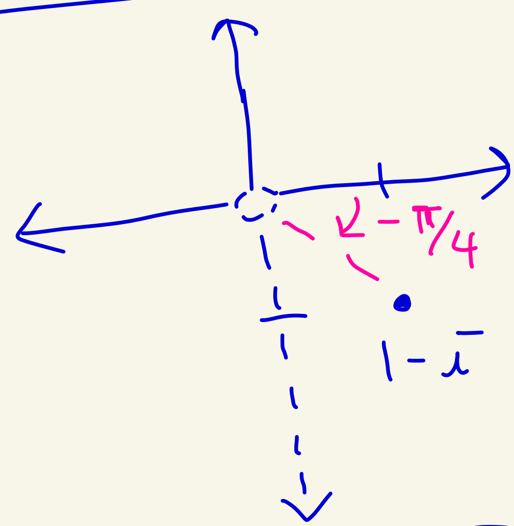
$\boxed{\text{and}}$

$$e^{(1+i)\log(1-i)} = e^{(1+i)\left[\ln|1-i| + i\arg(1-i)\right]}$$

$$= e^{(1+i)\left[\ln\sqrt{2} + i\left(-\pi/4\right)\right]}$$

$$= e^{\ln(\sqrt{2}) - i\frac{\pi}{4} + i\ln\sqrt{2} + \frac{\pi}{4}}$$

$$= e^{\ln(\sqrt{2}) + \frac{\pi}{4}}\left[\cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right]$$



$$\arg(1-i) = \frac{-\pi}{4}$$

$$-\frac{\pi}{2} < \frac{-\pi}{4} < \frac{3\pi}{2}$$

$$|1-i| = \sqrt{2}$$

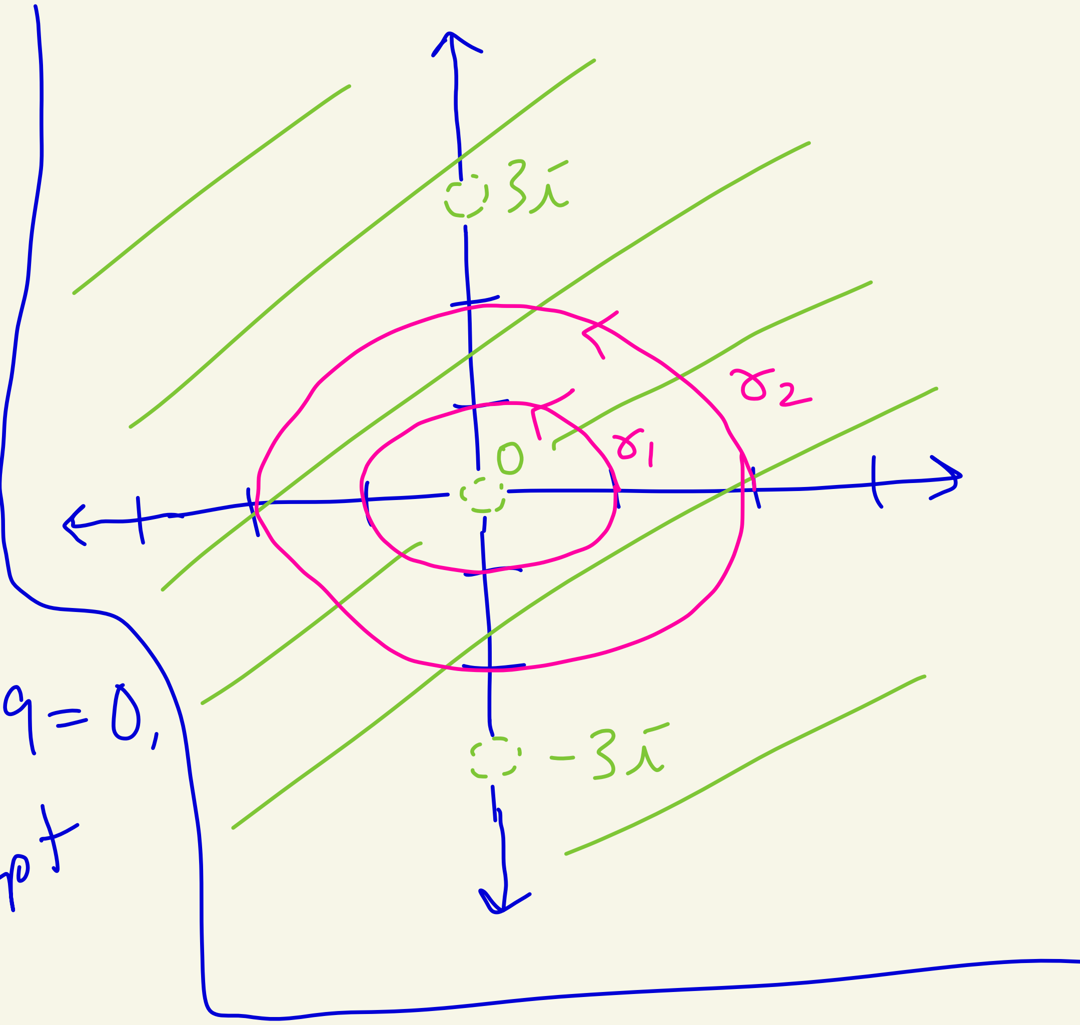
Thus,

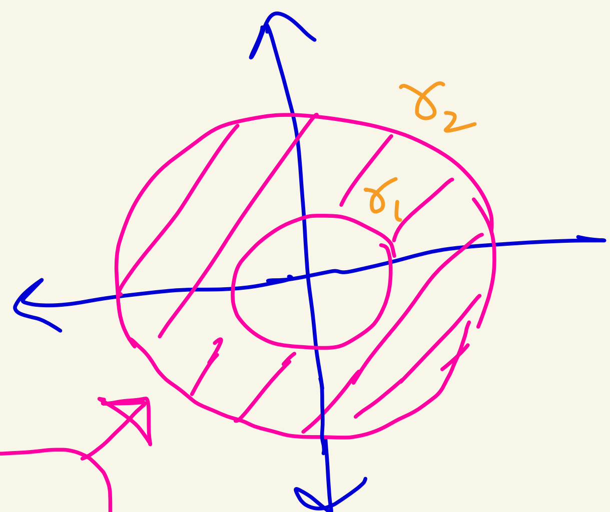
$$\int_\gamma z^i \, dz = \frac{1}{i+1}\Big[-e^{-\pi} - e^{\ln(\sqrt{2}) + \frac{\pi}{4}}\cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) - i e^{\ln\sqrt{2} + \frac{\pi}{4}}\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\Big]$$

②

Note that $f(z) = \dfrac{1}{z^{10}(z^2+9)}$ is analytic except when $z^{10} = 0$ or $z^2+9 = 0$,

That is, except when $z = 0$ or $z = \pm 3i$.





Since f is analytic on the closed set consisting of γ_1 and γ_2 and the points between them, we have that

$$\int_{\gamma_1} \frac{dz}{z^{10}(z^2+9)} = \int_{\gamma_2} \frac{dz}{z^{10}(z^2+9)}.$$