A Short Proof for Chen's Alternative Kneser Coloring Lemma

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July 9, 2012

Abstract

We give a short proof for Chen's Alternative Kneser Coloring Lemma. This leads to a short proof for the Johnson-Holroyd-Stahl conjecture that Kneser graphs have their circular chromatic numbers equal to their chromatic numbers.

1 Introduction

Suppose G is a graph and $p \geq q \geq 1$ are integers. A (p,q)-*coloring* of G is a mapping $c: V(G) \to \{0,1,\ldots,p-1\}$ such that $q \leq |f(x)-f(y)| \leq p-q$ for every edge xy of G. A graph is (p,q)-*colorable* if it admits a (p,q)-coloring. The *circular chromatic number* of G is

$$\chi_c(G) = \inf\{p/q\colon\ G \text{ is } (p,q)\text{-colorable}\}.$$

It is well-known [9] that for any graph G, $\chi(G)-1 < \chi_c(G) \leq \chi(G)$. The question as which graphs G satisfy the equality $\chi_c(G) = \chi(G)$ has received considerable attention.

Given positive integers $n \geq 2k$, the *Kneser graph* $\mathrm{KG}(n,k)$ has vertex set $\binom{[n]}{k}$, i.e., all k-subsets of $[n] = \{1,2,\ldots,n\}$, in which two vertices A and B are adjacent if $A \cap B = \emptyset$. Coloring of Kneser graphs has been a fascinating subject in graph theory. In proving Kneser's conjecture that $\chi(\mathrm{KG}(n,k)) = n-2k+2$, Lovász [6] initiated the application of algebraic topology to graph coloring. Since then, this method has became an important tool with wide applications in combinatorics.

Johnson, Holroyd and Stahl [5] first studied the circular chromatic number of Kneser graphs, and conjectured that the equality $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$ always holds. This conjecture has received a lot of attention. Hajiabolhassan and Zhu [4] proved that for a fixed k, if n is sufficiently large, then $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. Meunier [7] and Simonyi and Tardos [8] proved independently that if n is even then $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. The proof in [4] is combinatorial, and the proofs in [7, 8] use Fan's Lemma from algebraic topology. Nevertheless, both proofs also apply to Schrijver graphs $\mathrm{SG}(n,k)$ (subgraphs of $\mathrm{KG}(n,K)$ induced by stable

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k-subsets as vertices). On the other hand, it is known [8] that if n is odd and is not much bigger than $2k$, then $\chi_c(\mathrm{SG}(n,k)) \neq \chi(\mathrm{SG}(n,k))$. So it seemed not of much hope to apply these methods to completely prove the Johnson-Holroyd-Stahl conjecture.

However, recently Chen [1] completely proved the Johnson-Holroyd-Stahl conjecture by using Fan's Lemma in an innovative way. A key step in Chen's proof is to prove the *Alternative Kneser Coloring Lemma*. Assume $K_{q,q}$ is a complete bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$. Denote by $K^*_{q,q}$ the graph obtained from $K_{q,q}$ by deleting the edges of a perfect matching, say by deleting the edges $x_i y_i$ $(i = 1, 2, \ldots, q)$. Assume $K^*_{q,q}$ is a subgraph G and c is a q-coloring of G. We say $K^*_{q,q}$ is *colorful* with respect to c if $c(x_i) = c(y_i)$. Observe that if $K^*_{q,q}$ is colorful with respect to a q-coloring c, then $c(x_i) \neq c(x_j)$ for $i \neq j$, and hence we may assume that $c(x_i) = c(y_i) = i$ for $i = 1, 2, \ldots, q$.

Lemma 1 (Alternative Kneser Coloring Lemma [1]) *Any proper* $(n-2k+2)$*-coloring of* $\mathrm{KG}(n,k)$ *contains a colorful copy of* $K^*_{n-2k+2,n-2k+2}$.

Note that Lovász's result is equivalent to say that for every $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$, each color class is non-empty. Chen's Alternative Kneser Coloring Lemma reveals a more delicate structure of $(n-2k+2)$-colorings for $\mathrm{KG}(n,k)$. Besides its application to the determination of the circular chromatic number of Kneser graphs, the lemma is interesting by itself. For example, it provides a positive answer to a question asked in [3]: Every optimal coloring of a Kneser graph contains a subgraph H such that the close neighborhood $N_H[v]$ of each vertex of H uses all the colors.

Chen's proof of Lemma 1 is rather complicated. In this article, we give a shorter proof for this result. Before presenting it, for completeness, we show how Lemma 1 is used to settle the Johnson-Holroyd-Stahl conjecture. (A simple proof of this implication is also contained in [1] and [3].)

Lemma 2 *If* G *is* q*-colorable and every* q*-coloring of* G *contains a colorful copy of* $K^*_{q,q}$*, then* $\chi_c(G) = \chi(G) = q$.

Proof. For a q-coloring c of G, a cycle $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is called *tight* if $c(v_{i+1}) \equiv c(v_i) + 1 \pmod{q}$ for $i = 0, 1, \ldots, n-1$, where the indices of the vertices are modulo n. It is known [9] that $\chi_c(G) = q$ if and only if G is q-colorable and every q-coloring of G has a tight cycle. The assumption of Lemma 2 implies that every q-coloring c of G has a tight cycle. Assume a colorful copy of $K^*_{q,q}$ with respect to c has partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, with $c(x_i) = c(y_i) = i$ for $i = 1, 2, \ldots, q$. If q is even, then $(x_1, y_2, x_3, y_4, \ldots, x_{q-1}, y_q, x_1)$ is a tight cycle. If q is odd, then $(x_1, y_2, x_3, y_4, \ldots, y_{q-1}, x_q, y_1, x_2, y_3, x_4, \ldots, x_{q-1}, y_q, x_1)$ is a tight cycle. Thus, $\chi_c(G) = q$. ■

The Johnson-Holroyd-Stahl conjecture is an immediate consequence of Lemmas 1 and 2.

2 Proof of Alternative Kneser Coloring Lemma

We use Fan's Lemma to prove Chen's Alternative Kneser Coloring Lemma. Let n be a positive integer and let $[-1,1]^n = \{\mathbf{x} \in R^n : ||\mathbf{x}||_\infty \leq 1\}$ be the *n-dimensional cube*. The *barycentric subdivision* of $[-1,1]^n$, denoted by $\mathrm{sd}([-1,1]^n)$, is the simplicial complex whose vertices are points in $[-1,1]^n$ with each coordinate $0, 1$ or -1. A set of vertices form a simplex if the vertices can be ordered as $v_1, v_2, \dots, v_t$ so that for $i = 1, 2, \dots, t-1$, if a coordinate of v_i is 1 (or -1, respectively) then the corresponding coordinate of v_{i+1} is also 1 (or -1, respectively). The simplicial complex $\mathrm{sd}([-1,1]^n)$ is a triangulation of $[-1,1]^n$. The boundary of $\mathrm{sd}([-1,1]^n)$, denoted by $\partial(\mathrm{sd}([-1,1]^n))$, is a triangulation of the $(n-1)$-dimensional sphere S^{n-1}. Each vertex in $\partial(\mathrm{sd}([-1,1]^n))$ is a vector in $\{-1,1,0\}^n \setminus \{0\}^n$. We denote such a vector by a *signed set* X, which is a pair $X = (X^+, X^-)$ of disjoint subsets $X^+, X^- \subseteq [n]$, defined as $X^+ = \{i \colon X(i) = 1\}$ and $X^- = \{i \colon X(i) = -1\}$. Let $|X| = |X^+| + |X^-|$. We write $X \leq Y$ if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$, and write $X < Y$ if $X \leq Y$ and $X \neq Y$. Thus a simplex in $\partial(\mathrm{sd}([-1,1]^n))$ is a sequence of signed sets $\emptyset \neq X_1 < X_2 < \dots < X_t$.

An n-labeling of $\partial(\mathrm{sd}([-1,1]^n))$ is a mapping $\lambda \colon \{-1,1,0\}^n \setminus \{0\}^n \to \{\pm 1, \pm 2, \dots, \pm n\}$. An n-labeling λ is *antipodal* if $\lambda(-X) = -\lambda(X)$ for all X. A *complementary edge* with respect to λ is a pair of signed sets X, Y such that $X < Y$ and $\lambda(X) = -\lambda(Y)$. A simplex $X_1 < X_2 < \dots < X_n$ is a *positive alternating* $(n-1)$*-simplex* with respect to λ if $\{\lambda(X_1), \lambda(X_2), \dots, \lambda(X_n)\} = \{1, -2, \dots, (-1)^{n-1}n\}$. The following is a special case of Fan's Lemma.

Octahedral Fan's Lemma [2] *If λ is an antipodal n-labeling of the vertices of $\partial(\mathrm{sd}([-1,1]^n))$ without complementary edges, then the number of positive alternating $(n-1)$-simplices is odd.*

To apply Fan's Lemma, we shall associate to each proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$ with a labeling for the vertices of $\partial(\mathrm{sd}([-1,1]^n))$. Chen's proof of the Alternative Kneser Coloring Lemma also uses this approach. The difference between the two proofs is the labelings associated to the colorings of $\mathrm{KG}(n,k)$. Chen's labeling is the composition of two functions, including a rather complicated one, while the labeling we use is direct and simple.

Assume c is a proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$, using colors from the set $\{2k-1, 2k, \dots, n\}$. For a subset S of $[n]$ with $|S| \geq k$, let

$$c(S) = \max\{c(A) \colon A \subseteq S, |A| = k\}.$$

Let $\prec$ be an arbitrary linear ordering on subsets of $[n]$ such that $X \prec Y$ implies $|X| \leq |Y|$. Let $\lambda \colon \{-1,1,0\}^n \setminus \{0\}^n \to \{\pm 1, \pm 2, \dots, \pm n\}$ be defined as follows:

$$\lambda(X) = \begin{cases} |X|, & \text{if } |X| \leq 2k-2 \text{ and } X^- \prec X^+; \\ -|X|, & \text{if } |X| \leq 2k-2 \text{ and } X^+ \prec X^-; \\ c(X^+), & \text{if } |X| \geq 2k-1 \text{ and } X^- \prec X^+; \\ -c(X^-), & \text{if } |X| \geq 2k-1 \text{ and } X^+ \prec X^-. \end{cases}$$

It is obvious that λ is antipodal. It is also easy to verify that there are no complementary edges. Indeed, if $X < Y$ and $\lambda(X) = -\lambda(Y)$, then by definition of λ, it must be the case that $|X|, |Y| \geq 2k-1$. Assume $\lambda(X) > 0$ (the other case is symmetric). Then there exist $X' \subseteq X^+ \subseteq Y^+$ and $Y' \subseteq Y^-$ such that $|X'| = |Y'| = k$ and $c(X') = c(Y')$. However, $Y^+ \cap Y^- = \emptyset$, implying that $X'Y'$ is an edge of $\mathrm{KG}(n,k)$, a contradiction. Thus, by Fan's Lemma, there are an odd number of positive alternating $(n-1)$-simplices.

Assume $X_1 < X_2 < \dots < X_n$ is a positive alternating $(n-1)$-simplex with respect to λ. Since $1 \leq |X_1| < |X_2| < \dots < |X_n| \leq n$, one has $|X_i| = i$ for $1 \leq i \leq n$.

Claim 1. Let $X_0 = (\emptyset, \emptyset)$. For any index $1 \leq i \leq n$, either $|X_i^+| = |X_{i-1}^+| + 1$, $X_{i-1}^- = X_i^- \prec X_i^+$ and $\lambda(X_i) > 0$, or else $|X_i^-| = |X_{i-1}^-| + 1$, $X_{i-1}^+ = X_i^+ \prec X_i^-$ and $\lambda(X_i) < 0$.

Proof. For $1 \le i \le 2k-2$, it follows from the definitions of λ and the positive alternating $(n-1)$-simplices that $\lambda(X_i) = (-1)^{i-1}i$, and hence if i is odd, then $|X_i^+| = |X_{i-1}^+| + 1$ and $X_{i-1}^- = X_i^- \prec X_i^+$; if i is even, then $|X_i^-| = |X_{i-1}^-| + 1$, $X_{i-1}^+ = X_i^+ \prec X_i^-$. In particular, $|X_{2k-2}^+| = |X_{2k-2}^-| = k-1$.

Assume $2k-1 \le i \le n$. Since $X_{i-1} < X_i$ and $|X_i| = |X_{i-1}| + 1$, we know that either $|X_i^+| = |X_{i-1}^+| + 1$ and $X_{i-1}^- = X_i^-$, or else $|X_i^-| = |X_{i-1}^-| + 1$ and $X_{i-1}^+ = X_i^+$. Assume $|X_i^+| = |X_{i-1}^+| + 1$ and $X_{i-1}^- = X_i^-$ (the other case is symmetric). Assume to the contrary of the claim that $X_i^+ \prec X_i^-$. Then $|X_i^+| \le |X_i^-|$ and so $|X_{i-1}^+| < |X_{i-1}^-|$ which gives $X_{i-1}^+ \prec X_{i-1}^-$ and $i-1 \ne 2k-2$. Hence, $\lambda(X_i) = -c(X_i^-) = -c(X_{i-1}^-) = \lambda(X_{i-1})$, contradicting the fact that $\lambda(X_r) \ne \lambda(X_s)$ for $r \ne s$. □

Since $\lceil n/2 \rceil$ of the labels $\lambda(X_i)$'s are positive and $\lfloor n/2 \rfloor$ of them are negative, it follows from Claim 1 that $|X_n^+| = \lceil n/2 \rceil$ and $|X_n^-| = \lfloor n/2 \rfloor$.

Claim 2. For any index $1 \le i \le n$, it holds that $-1 \le |X_i^+| - |X_i^-| \le 1$.

Proof. By symmetry, it is enough to show that $|X_i^+| - |X_i^-| \le 1$. Assume to the contrary that $|X_i^+| - |X_i^-| \ge 2$ for some i. Since $|X_n^+| - |X_n^-| \le 1$, there is an index j such that $|X_{j+1}^+| - |X_{j+1}^-| \le 1 < 2 \le |X_j^+| - |X_j^-|$. Hence $|X_{j+1}^-| = |X_j^-| + 1$. By Claim 1, $X_{j+1}^+ \prec X_{j+1}^-$ and so $|X_{j+1}^+| \le |X_{j+1}^-|$, which is impossible as $|X_j^+| - |X_j^-| \ge 2$. □

It follows from Claim 2 that $|X_{2j}^+| = |X_{2j}^-| = j$ for $1 \le j \le n/2$. So we may denote $[n] = \{a_1, a_2, \ldots, a_n\}$ where $X_{2j}^+ = \{a_1, a_3, \ldots, a_{2j-1}\}$ and $X_{2j}^- = \{a_2, a_4, \ldots, a_{2j}\}$. The signed set X_{2j-1} can be either (X_{2j}^+, X_{2j-2}^-) or (X_{2j-2}^+, X_{2j}^-).

As observed above, $\lambda(X_i) = (-1)^{i-1}i$ for $1 \le i \le 2k-2$. For $2k-1 \le i \le n$, since $\{\lambda(X_{2k-1}), \lambda(X_{2k}), \ldots, \lambda(X_n)\} = \{2k-1, -2k, \ldots, (-1)^{n-1}n\}$, by the monotonicity of c,

$$c(\{a_1, a_3, \ldots, a_i\}) = i \text{ for odd } i; \text{ and } c(\{a_2, a_4, \ldots, a_i\}) = i \text{ for even } i.$$

Let $\Gamma = \{X \in \{+,-,0\}^n : |X^+| = |X^-| = k-1\}$. As noted above, each positive alternating $(n-1)$-simplex contains exactly one vertex in Γ. For $X \in \Gamma$, let $\alpha(X, \lambda)$ be the number of positive alternating $(n-1)$-simplices containing vertex X. By Fan's Lemma, $\Sigma_{X \in \Gamma} \alpha(X, \lambda)$ is odd. Hence there exists $Z \in \Gamma$ such that $\alpha(Z, \lambda)$ is odd. In particular, there exists a positive alternating $(n-1)$-simplex $X_1 < X_2 < \cdots < X_n$ with respect to λ, with $Z = X_{2k-2}$. For this Z, define $\lambda' : \{+,-,0\}^n \setminus \{0\}^n \to \{\pm 1, \pm 2, \ldots, \pm n\}$ by:

$$\lambda'(X) = \begin{cases} -\lambda(X), & \text{if } X \in \{Z, -Z\}; \\ \lambda(X), & \text{otherwise.} \end{cases}$$

Then λ' is antipodal without complementary edges. By Fan's Lemma, there are an odd number of positive alternating $(n-1)$-simplices with respect to λ'. Since $\alpha(X, \lambda') = \alpha(X, \lambda)$ for $X \in \Gamma \setminus \{Z, -Z\}$, we conclude that

$$\alpha(Z, \lambda) + \alpha(-Z, \lambda) \equiv \alpha(Z, \lambda') + \alpha(-Z, \lambda') \pmod 2.$$

Since $\lambda(Z) = -(2k-2)$ and so $\lambda(-Z) = 2k-2 = \lambda'(Z)$, we know that $\alpha(-Z, \lambda) = \alpha(Z, \lambda') = 0$. Thus, $\alpha(-Z, \lambda') \equiv \alpha(Z, \lambda) \equiv 1 \pmod 2$. So there exists a positive alternating $(n-1)$-simplex $Y_1 < Y_2 < \cdots < Y_n$ with respect to λ', where $Y_{2k-2} = -Z$. Similar to the discussion for λ, we may denote $[n] = \{b_1, b_2, \ldots, b_n\}$ where $Y_{2j}^+ = \{b_1, b_3, \ldots, b_{2j-1}\}$ and $Y_{2j}^- = \{b_2, b_4, \ldots, b_{2j}\}$. The signed set Y_{2j-1} can be either (Y_{2j}^+, Y_{2j-2}^-) or (Y_{2j-2}^+, Y_{2j}^-), where $Y_0^+ = Y_0^- = \emptyset$. Also, for $2k-1 \le i \le n$, $c(\{b_1, b_3, \ldots, b_i\}) = i$ for odd i; and $c(\{b_2, b_4, \ldots, b_i\}) = i$ for even i.

Let $Z = (S, T)$. Then $X_{2k-2} = (S, T)$ and $Y_{2k-2} = (T, S)$. Consequently, for $2k-1 \le i \le n$,

$$c(S \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_i\}) = c(T \cup \{b_{2k-1}, b_{2k+1}, \ldots, b_i\}) = i \text{ for odd } i\text{; and}$$
$$c(T \cup \{a_{2k}, a_{2k+2}, \ldots, a_i\}) = c(S \cup \{b_{2k}, b_{2k+2}, \ldots, b_i\}) = i \text{ for even } i.$$

Claim 3. For any index $2k-1 \le i \le n$, it holds that $a_i = b_i$ and $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$.

Proof. We prove by induction on i. If $i = 2k-1$, since $c(S \cup \{a_{2k-1}\}) = c(T \cup \{b_{2k-1}\}) = 2k-1$, so $S \cup \{a_{2k-1}\}$ and $T \cup \{b_{2k-1}\}$ are not adjacent, implying $a_{2k-1} = b_{2k-1}$. Assume $i \ge 2k$ and the claim is true for $i' < i$. If i is odd, then since for all $2k-1 \le j < i$, $S \cup \{a_i\}$ and $T \cup \{a_j\}$ are adjacent, so $c(S \cup \{a_i\}) \ne c(T \cup \{a_j\}) = j$ for $2k-1 \le j < i$. Because $c(S \cup \{a_i\}) \le c(S \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_i\}) = i$, we conclude that $c(S \cup \{a_i\}) = i$. Similarly, $c(T \cup \{b_i\}) = i$. As $c(S \cup \{a_i\}) = c(T \cup \{b_i\})$, so $S \cup \{a_i\}$ and $T \cup \{b_i\}$ are not adjacent. Hence $a_i = b_i$. If i is even, by the same argument, we have $c(S \cup \{b_i\}) = c(T \cup \{a_i\}) = i$, which implies that $a_i = b_i$. This completes the proof of the claim. □

The subgraph of $\mathrm{KG}(n, k)$ induced by the vertices $\{S \cup \{a_i\}, T \cup \{a_i\} \colon 2k-1 \le i \le n\}$ is a colorful copy of $K^*_{n-2k+2,n-2k+2}$. This completes the proof of Chen's Alternative Kneser Coloring Lemma.

Acknowledgment. We are grateful to the two anonymous referees for their quick reports with constructive suggestions.

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