THE LENGTH OF NOETHERIAN MODULES

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ABSTRACT. We define an ordinal valued length for Noetherian modules which extends the usual definition of composition series length for finite length modules. Though originally defined by Gulliksen [1] in the 1970s, this extension has been seldom used in subsequent research. Despite this neglect, we will show that the ordinal valued length is a quite natural measure of the size of a Noetherian module, and has advantages over more familiar measures such as uniform dimension, Krull dimension, and reduced rank. We will also demonstrate how some familiar properties of left Noetherian rings can be proved efficiently using length and the arithmetic properties of ordinal numbers.

1. INTRODUCTION

In the early 1970s, T. H. Gulliksen [1] showed how the definition of composition series length, defined only for finite length modules, could be extended to give an ordinal valued length for any Noetherian module. During the same time period, various people (e.g. [2], [3]) defined another ordinal valued measure for modules, the Krull dimension. This second line of research culminated in the article *Krull Dimension* by R. Gordon and J. C. Robson [4]. Since that time, the paper by Gulliksen has been rarely cited ([5], [6], [7], [8], [9]), whereas the Gordon and Robson article has been cited over 175 times.

It is the main purpose of this paper to point out that, contrary to what one might expect from the above, for a Noetherian module B, its ordinal valued length, $\operatorname{len} B$, is a more natural measure of its size than its Krull dimension, $\operatorname{Kdim} B$. Moreover $\operatorname{len} B$ contains more information about the size of B.

Length and Krull dimension are really measures of the size of the lattice $\mathcal{L}(B)$ of all submodules of B ordered by inclusion. We will write $\mathcal{L}^{\circ}(B)$ for the set of submodules of B ordered by reverse inclusion, that is, the dual of $\mathcal{L}(B)$. A module B is Noetherian if and only if $\mathcal{L}(B)$ is Noetherian if and only if $\mathcal{L}^{\circ}(B)$ is Artinian.

Suppose B is a Noetherian uniserial module, meaning that $\mathcal{L}(B)$ is Noetherian and totally ordered. Then $\mathcal{L}^\circ(B)$ is a well ordered set with maximum element 0. Following the convention of counting the gaps rather than the modules in a finite chain, we define the length of B, $\operatorname{len} B$, to be the ordinal represented by $\mathcal{L}^\circ(B) \setminus \{0\}$. Using this definition and the arithmetic of ordinal numbers we can then prove various properties of uniserial modules.

For a simple example, we notice that if $0 \to A \to B \to C \to 0$ is exact, then $\operatorname{len} B = \operatorname{len} C + \operatorname{len} A$. Consider the case when $\operatorname{len} B = \operatorname{len} C$. Here $\operatorname{len} B = \operatorname{len} B + \operatorname{len} A$, and, since ordinal addition is cancellative on the left, we get $\operatorname{len} A = 0$ and $A = 0$. Expressed differently, this says that a homomorphism ψ on B is injective if (and only if) $\operatorname{len} B = \operatorname{len} \psi(B)$. As a special case, any surjective endomorphism of B is injective.

The definition of length in this paper extends the above definition for uniserial modules to all Noetherian modules. It is natural because there is really only one possible way of making this extension: In short, for a Noetherian module B, we define $\operatorname{len} B = \lambda(0)$ where λ is the smallest possible strictly decreasing function from $\mathcal{L}(B)$ into the class of ordinal numbers. This is equivalent to Gulliksen's original definition. The function λ can also be defined inductively as follows: First set $\lambda(B) = 0$. Suppose, for an ordinal α, we have already identified those submodules B' of B such that $\lambda(B') < \alpha$. Then $\lambda(B') = \alpha$ if and only if B' is maximal among those submodules of B on which λ has not yet been defined.

Once again ordinal arithmetic comes into play. We will show (4.1) that if $0 \to A \to B \to C \to 0$ is an exact sequence of Noetherian modules then $\operatorname{len} C + \operatorname{len} A \leq \operatorname{len} B \leq \operatorname{len} C \oplus \operatorname{len} A$. Here $\oplus$ is the "natural" sum on ordinals (2.7). We have already noted that if B is uniserial then $\operatorname{len} C + \operatorname{len} A = \operatorname{len} B$. The other extreme case occurs if the sequence splits: If $B \cong A \oplus C$, then $\operatorname{len} B = \operatorname{len} C \oplus \operatorname{len} A$. Since $\oplus$ is a cancellative operation (2.8) on ordinals, we have immediately that $A \oplus C \cong B \oplus C$ implies that $\operatorname{len} A = \operatorname{len} B$ for Noetherian modules A, B and C.

The relationship between $\operatorname{len} B$ and $\operatorname{Kdim} B$ is a simple one: If B is nonzero, then the length of B can be written uniquely in the **long normal form** $\operatorname{len} B = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are ordinals. Then $\operatorname{Kdim} B = \gamma_1$. In fact, the (finite number of) possible values of $\operatorname{len} B'$ for a submodule $B' \leq B$, are determined by $\operatorname{len} B$. In particular, we have $\operatorname{Kdim} B' \in \{-1, \gamma_1, \gamma_2, \ldots, \gamma_n\}$ (4.6). Thus $\operatorname{len} B$ contains a lot more information about B than $\operatorname{Kdim} B$.

Much can be proved about Noetherian modules using only ordinal arithmetic and the above rule about short exact sequences. For more complicated theorems, the existence of the long normal form for ordinals permits proofs which are finite inductions on the number of terms in such a form. In the final section of this paper we demonstrate this technique in proving some familiar properties of left Noetherian rings.

2. The Length of a Partially Ordered Set

As indicated in the introduction, both the length and Krull dimension of a Noetherian module B, are measures of the size of the lattice of submodules of B. Thus it is convenient to define these concepts first for lattices, or indeed for partially ordered sets.

The key concept in this section is the length function on a partially ordered set $\mathcal{L}$. This is a certain function from $\mathcal{L}$ into the class of ordinal numbers, **Ord**. We will find it convenient at first to define length functions, not just on partially ordered sets, but also on partially ordered classes such as **Ord** itself and **Ord** $\times$ **Ord**. Thus we phrase our definitions below in this generality. Note that **Ord** only barely fails to be a set in the sense that, for any ordinal α, the class of elements of **Ord** which are less than α is a set.

If $\mathcal{L}$ is a partially ordered class and $x, y \in \mathcal{L}$, we will use the following notation:

$$\{\leq x\} = \{z \in \mathcal{L} \mid z \leq x\} \qquad [x, y] = \{z \in \mathcal{L} \mid x \leq z \leq y\}.$$

If $\psi\colon \mathcal{K} \to \mathcal{L}$ is a function between partially ordered classes, then

- ψ is **increasing** if $x \leq y$ in $\mathcal{K}$ implies $\psi(x) \leq \psi(y)$ in $\mathcal{L}$.
- ψ is an **isomorphism** (and $\mathcal{K} \cong \mathcal{L}$) if ψ is a bijection such that ψ and ψ^{-1} are increasing. Note that an increasing bijection may not be an isomorphism.
- ψ is **strictly increasing** if $x < y$ in $\mathcal{K}$ implies $\psi(x) < \psi(y)$ in $\mathcal{L}$.
- ψ is **exact** if $\{\leq \psi(x)\} \subseteq \psi(\{\leq x\})$ for all $x \in \mathcal{K}$.

One can easily check that if $\psi_1\colon \mathcal{L} \to \mathcal{M}$ and $\psi_2\colon \mathcal{K} \to \mathcal{L}$ are increasing (strictly increasing, exact) functions, then so is $\psi_1 \circ \psi_2\colon \mathcal{K} \to \mathcal{M}$.

We will write $\mathcal{L}_1 \times \mathcal{L}_2$ for the Cartesian product of two partially ordered classes $\mathcal{L}_1$ and $\mathcal{L}_2$, with order given by

$$(x_1, x_2) \leq (y_1, y_2) \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2).$$

Notice that $\{\leq (x_1, x_2)\} = \{\leq x_1\} \times \{\leq x_2\}$. It is easy to show that $\mathcal{L}_1 \times \mathcal{L}_2 \cong \mathcal{L}_2 \times \mathcal{L}_1$ and $(\mathcal{L}_1 \times \mathcal{L}_2) \times \mathcal{L}_3 \cong \mathcal{L}_1 \times (\mathcal{L}_2 \times \mathcal{L}_3)$.

If $\psi_1: \mathcal{K}_1 \to \mathcal{L}_1$ and $\psi_2: \mathcal{K}_2 \to \mathcal{L}_2$ are maps between partially ordered classes, then we will write $\psi_1 \times \psi_2$ for the map from $\mathcal{K}_1 \times \mathcal{K}_2$ to $\mathcal{L}_1 \times \mathcal{L}_2$ given by $(\psi_1 \times \psi_2)(x_1, x_2) = (\psi_1(x_1), \psi_2(x_2))$. One can easily check that if ψ_1 and ψ_2 are increasing (strictly increasing, exact), then so is $\psi_1 \times \psi_2$.

A partially ordered class $\mathcal{L}$ is **Artinian** (**Noetherian**) if every nonempty subclass has a minimal (maximal) element (equivalently, every strictly decreasing (increasing) sequence in $\mathcal{L}$ is finite.) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are Artinian (Noetherian) partially ordered classes, then so is $\mathcal{L}_1 \times \mathcal{L}_2$.

Other notation: $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers, and $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$ is the set of nonnegative integers.

The theorems in this paper depend heavily on the arithmetic of the ordinal numbers. For the details of ordinal arithmetic see W. Sierpinski, *Cardinal and Ordinal Numbers* [10] or M. D. Potter, *Sets, An Introduction* [11]. We collect here a few of those facts that are relevant:

We will use lowercase Greek letters for ordinal numbers. The smallest infinite ordinal is written ω.

- Ordinal addition is associative but not commutative. For example, $\omega + 1 \neq 1 + \omega = \omega$.
- Ordinal addition is cancellative on the left: $\alpha + \beta = \alpha + \gamma \implies \beta = \gamma$. Also $\alpha + \beta \leq \alpha + \gamma \implies \beta \leq \gamma$.
- For a fixed ordinal α, the map from **Ord** to **Ord** given by $\beta \mapsto \alpha + \beta$ is strictly increasing.
- If $\alpha \leq \beta$, then $\beta - \alpha$ is the unique ordinal γ such that $\beta = \alpha + \gamma$, hence $\beta = \alpha + (\beta - \alpha)$. For any $\alpha, \beta \in$ **Ord**, we have $\beta = (\alpha + \beta) - \alpha$.
- For a fixed ordinal α, the map from $\{\beta \in \mathbf{Ord} \mid \alpha \leq \beta\}$ to **Ord** given by $\beta \mapsto \beta - \alpha$ is strictly increasing.
- $\alpha n = \alpha + \alpha + \ldots + \alpha$ (n times) when $n \in \mathbb{N}$. Note: $3\omega = \omega \neq \omega 3$.

The most important property of ordinal numbers is that any nonzero ordinal α can be expressed uniquely in **long normal form**

$$\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$$

where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are ordinals. By collecting together terms which have identical exponents, this same form can be written

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$$

where now $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ and $n_1, n_2, \dots, n_n \in \mathbb{N}$. This we will call the **short normal form** for α.

Certain parameters in these normal forms will have an important role in our discussion of Noetherian modules in later sections:

Definition 2.1. *For a nonzero ordinal* $\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$ *in short normal form we define the* **Krull dimension** *of* α *by* $\operatorname{Kdim} \alpha = \gamma_1$ *and the* **Krull rank** *of* α *by* $\operatorname{Krank} \alpha = \sum_{i=1}^{n} n_i$. *For* $i = 1, 2, \ldots, n$, *the number* n_i *will be called the* γ_i**-length** *of* α, *written* $\operatorname{len}_{\gamma_i} \alpha$. *For an ordinal* γ *not in* $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ *we define* $\operatorname{len}_\gamma \alpha = 0$. *By convention* $\operatorname{Kdim} 0 = -1$, $\operatorname{Krank} 0 = 0$, *and* $\operatorname{len}_\gamma 0 = 0$.

If $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form, then $\operatorname{Kdim} \alpha = \gamma_1$ and $\operatorname{Krank} \alpha = n$.

Most of the arithmetic properties of ordinals we will need are consequences of the fact that to add two ordinals in normal form one needs only the associativity of addition and the rule that $\omega^\gamma + \omega^\delta = \omega^\delta$ if $\gamma < \delta$. For example, $(\omega^\omega + \omega^3 + \omega 2 + 1) + (\omega^3 + \omega) = \omega^\omega + \omega^3 2 + \omega$. Using this rule one can readily prove the following:

Lemma 2.2. *Suppose* $\alpha, \beta, \gamma \in \mathbf{Ord}$ *with* $\alpha > 0$ *and* $m, n \in \mathbb{Z}^+$.

(1) $\beta + \alpha \leq \alpha \iff \beta + \alpha = \alpha \iff \operatorname{Kdim} \beta < \operatorname{Kdim} \alpha$
(2) $\alpha = \omega^\gamma$ *for some* $\gamma \in \mathbf{Ord} \iff \beta + \alpha = \alpha$ *for all* $\beta < \alpha$
$\iff \operatorname{Kdim} \beta < \operatorname{Kdim} \alpha$ *for all* $\beta < \alpha$
(3) $\beta + \omega^\gamma n < \omega^\gamma m \implies \beta < \omega^\gamma (m - n)$

The $<$ symbols are necessary in 3: If $\beta = \gamma = m = n = 1$, then $\beta + \omega^\gamma n \leq \omega^\gamma m$ but $\beta \nleq \omega^\gamma (m - n)$.

Now suppose we have a partially ordered class $\mathcal{L}$. We would like to say something about the size of $\mathcal{L}$ by considering certain "order preserving" functions from $\mathcal{L}$ into **Ord**.

Considering even the case when $\mathcal{L}$ is finite and totally ordered, it is easy to list some properties that such a function $\lambda \colon \mathcal{L} \to \mathbf{Ord}$ should have. For example, we would certainly want λ to be strictly increasing. We would not want λ to skip any ordinals unnecessarily, that is, if $\alpha < \lambda(x)$ for some $x \in \mathcal{L}$, then there ought to be some $y < x$ such that $\lambda(y) = \alpha$. Thus λ should be exact.

The key result of this section is that these two conditions suffice to specify a unique "order preserving" function from $\mathcal{L}$ into **Ord**.

Theorem 2.3. *Let* $\mathcal{L}$ *be a partially ordered class. If there exists at least one strictly increasing function from* $\mathcal{L}$ *to* **Ord** *then* $\mathcal{L}$ *is Artinian and there exists a unique function* $\lambda_{\mathcal{L}} \colon \mathcal{L} \to \mathbf{Ord}$ *satisfying the following equivalent conditions:*

(1) $\lambda_{\mathcal{L}}$ *is strictly increasing, and, if* $\lambda \colon \mathcal{L} \to \mathbf{Ord}$ *is a strictly increasing function, then* $\lambda_{\mathcal{L}}(x) \leq \lambda(x)$ *for all* $x \in \mathcal{L}$.
(2) *For all* $x \in \mathcal{L}$ *and* $\alpha \in \mathbf{Ord}$, $\lambda_{\mathcal{L}}(x) = \alpha$ *if and only if* x *is minimal in* $\mathcal{K}_\alpha = \{y \in \mathcal{L} \mid \lambda_{\mathcal{L}}(y) \geq \alpha\}$.

(3) $\lambda_{\mathcal{L}}$ *is strictly increasing and exact.*

Proof. Any strictly increasing function from $\mathcal{L}$ to **Ord**, maps infinite strictly decreasing sequences in $\mathcal{L}$ to infinite strictly decreasing sequences in **Ord**. Since no such sequences exist in **Ord**, there are no infinite strictly decreasing sequences in $\mathcal{L}$ either.

Define

$$\lambda_{\mathcal{L}}(x) = \min\{\lambda(x) \mid \lambda\colon \mathcal{L} \to \textbf{Ord} \text{ is strictly increasing}\}$$

for all $x \in \mathcal{L}$. Since we are assuming that at least one strictly increasing function exists, $\lambda_{\mathcal{L}}$ is well defined by this equation. If $x < y$ in $\mathcal{L}$, then there is some strictly increasing function $\lambda\colon \mathcal{L} \to \textbf{Ord}$ such that $\lambda(y) = \lambda_{\mathcal{L}}(y)$, so $\lambda_{\mathcal{L}}(x) \leq \lambda(x) < \lambda(y) = \lambda_{\mathcal{L}}(y)$. Thus $\lambda_{\mathcal{L}}$ is itself strictly increasing and satisfies condition 1. The uniqueness of $\lambda_{\mathcal{L}}$ is immediate from condition 1.

We now show the equivalence of 1, 2 and 3:

$1 \Rightarrow 2$: Suppose $\lambda_{\mathcal{L}}(x) = \alpha$. Then $x \in \mathcal{K}_\alpha$. Since $\lambda_{\mathcal{L}}$ is strictly increasing, for any $y < x$, we have $\lambda_{\mathcal{L}}(y) < \lambda_{\mathcal{L}}(x) = \alpha$ and so $y \notin \mathcal{K}_\alpha$. Hence x is minimal in $\mathcal{K}_\alpha$.

Now suppose x is minimal in $\mathcal{K}_\alpha$. In particular $\alpha \leq \lambda_{\mathcal{L}}(x)$. Define $\lambda\colon \mathcal{L} \to \textbf{Ord}$ by

$$\lambda(y) = \begin{cases} \lambda_{\mathcal{L}}(y) & y \neq x \\ \alpha & y = x \end{cases}$$

It is easy to check that λ is strictly increasing. From our hypothesis on $\lambda_{\mathcal{L}}$ we get $\lambda_{\mathcal{L}}(x) \leq \lambda(x) = \alpha$, and so $\lambda_{\mathcal{L}}(x) = \alpha$.

$2 \Rightarrow 3$: Suppose $y < x$ in $\mathcal{L}$ and $\lambda_{\mathcal{L}}(x) = \alpha$. Since x is minimal in $\mathcal{K}_\alpha$ we have $y \notin \mathcal{K}_\alpha$ and hence $\lambda_{\mathcal{L}}(y) < \alpha = \lambda_{\mathcal{L}}(x)$. Thus $\lambda_{\mathcal{L}}$ is strictly increasing.

For exactness we need to show $\{\leq \lambda_{\mathcal{L}}(y)\} \subseteq \lambda_{\mathcal{L}}(\{\leq y\})$ for all $y \in \mathcal{L}$. Suppose then we have $\alpha \leq \lambda_{\mathcal{L}}(y)$. Then $y \in \mathcal{K}_\alpha$, and, since $\mathcal{L}$ is Artinian, there is some $x \leq y$ which is minimal in $\mathcal{K}_\alpha$. By hypothesis, $\lambda_{\mathcal{L}}(x) = \alpha$ as desired.

$3 \Rightarrow 1$: Let $\lambda\colon \mathcal{L} \to \textbf{Ord}$ be strictly increasing. We will show that $\lambda_{\mathcal{L}}(x) \leq \lambda(x)$ for all $x \in \mathcal{L}$.

Suppose to the contrary that $\mathcal{K} = \{x \in \mathcal{L} \mid \lambda(x) < \lambda_{\mathcal{L}}(x)\}$ is nonempty. Let x be chosen in $\mathcal{K}$ so that $\lambda(x)$ is minimum in $\lambda(\mathcal{K})$.

For any $y < x$ we have $\lambda(y) < \lambda(x)$, so $y \notin \mathcal{K}$ and $\lambda_{\mathcal{L}}(y) \leq \lambda(y) < \lambda(x) < \lambda_{\mathcal{L}}(x)$. Thus we have an ordinal $\alpha = \lambda(x)$ such that $\alpha < \lambda_{\mathcal{L}}(x)$ but there is no $y < x$ with $\lambda_{\mathcal{L}}(y) = \alpha$. This contradicts exactness. □

Definition 2.4. *The function* $\lambda_{\mathcal{L}}\colon \mathcal{L} \to \textbf{Ord}$ *(when it exists) will be called the* **length function** *on* $\mathcal{L}$*. If* $\mathcal{L}$ *has a maximum element* $\top$*, then we define the* **length** *of* $\mathcal{L}$ *by* $\operatorname{len}\mathcal{L} = \lambda_{\mathcal{L}}(\top)$*. In addition, the* **Krull dimension**,

Krull rank *and* γ**-length** *of* $\mathcal{L}$ *are defined by* $\operatorname{Kdim}\mathcal{L} = \operatorname{Kdim}(\operatorname{len}\mathcal{L})$, $\operatorname{Krank}\mathcal{L} = \operatorname{Krank}(\operatorname{len}\mathcal{L})$ *and* $\operatorname{len}_\gamma \mathcal{L} = \operatorname{len}_\gamma(\operatorname{len}\mathcal{L})$.

Theorem 2.3(2) suggests that we could define $\lambda_{\mathcal{L}}$ inductively using the relationship

$\lambda_{\mathcal{L}}(x) = \alpha$ if and only if x is minimal in $\mathcal{K}_\alpha = \{y \in \mathcal{L} \mid \lambda_{\mathcal{L}}(y) \not< \alpha\}$.

Notice in particular that, from this definition, $\lambda_{\mathcal{L}}(x) = 0$ if and only if x is minimal in $\mathcal{L}$.

Certainly, if $\mathcal{L}$ has a length function then, from 2.3, this definition produces the length function of $\mathcal{L}$. If $\mathcal{L}$ is not known to have a length function then this definition will produce the length function of some (possibly empty) subclass of $\mathcal{L}$.

Theorem 2.5. *Let* $\mathcal{L}$ *be an Artinian partially ordered class. If* $\{\leq x\}$ *is a set for all* $x \in \mathcal{L}$*, then* $\mathcal{L}$ *has a length function.*

Proof. Let $\lambda_{\mathcal{L}}$ be defined inductively as above. Suppose $x \in \mathcal{L}$ is minimal among elements for which $\lambda_{\mathcal{L}}$ is undefined. Then for every $z < x$, $\lambda_{\mathcal{L}}(z)$ is defined. Let $\alpha = \sup\{\lambda_{\mathcal{L}}(z) + 1 \mid z < x\}$. This is well defined since any sub<u>set</u> of **Ord** has a supremum [12, Section 20]. It is then easy to show that x is minimal in $\mathcal{K}_\alpha = \{y \in \mathcal{L} \mid \lambda_{\mathcal{L}}(y) \not< \alpha\}$ and so $\lambda_{\mathcal{L}}(x) = \alpha$. This contradicts our assumption that $\lambda_{\mathcal{L}}(x)$ is undefined. Consequently, $\lambda_{\mathcal{L}}$ is defined on all of $\mathcal{L}$, and then by 2.3, $\lambda_{\mathcal{L}}$ is the length function of $\mathcal{L}$. □

From the proof of this theorem we notice that for any $x \in \mathcal{L}$,

$$\lambda_{\mathcal{L}}(x) = \sup\{\lambda_{\mathcal{L}}(z) + 1 \mid z < x\}.$$

This equation could also be used inductively to define $\lambda_{\mathcal{L}}$. In fact, this formula, written in terms of the lattice of submodules of a Noetherian module, is part of Gulliksen's original definition of the length of a Noetherian module.

Lemma 2.6. *Let* $\mathcal{L}$ *and* $\mathcal{K}$ *be partially ordered classes with length functions and maximum elements.*

(1) *For all* $x \in \mathcal{L}$, $\operatorname{len}\{\leq x\} = \lambda_{\mathcal{L}}(x)$.
(2) *For all* $x \leq y \in \mathcal{L}$, $\operatorname{len}\{\leq x\} + \operatorname{len}[x, y] \leq \operatorname{len}\{\leq y\}$.
(3) *For any ordinal* $\alpha \leq \operatorname{len}\mathcal{L}$*, there is some* $x \in \mathcal{L}$ *such that* $\lambda_{\mathcal{L}}(x) = \alpha$.
(4) *If* $\lambda : \mathcal{L} \to \mathcal{K}$ *is a strictly increasing function, then* $\operatorname{len}\mathcal{L} \leq \operatorname{len}\mathcal{K}$.

Proof. (1) It is easy to see that the restriction of $\lambda_{\mathcal{L}}$ to $\{\leq x\}$ is strictly increasing and exact, so by 2.3, $\lambda_{\mathcal{L}} = \lambda_{\{\leq x\}}$ on $\{\leq x\}$. In particular, $\lambda_{\mathcal{L}}(x) = \lambda_{\{\leq x\}}(x) = \operatorname{len}\{\leq x\}$.

(2) Define $\lambda\colon [x,y] \to \mathbf{Ord}$ by $\lambda(z) = \lambda_{\mathcal{L}}(z) - \lambda_{\mathcal{L}}(x)$. The function λ is strictly increasing, so

$$\operatorname{len}[x,y] \le \lambda(y) = \lambda_{\mathcal{L}}(y) - \lambda_{\mathcal{L}}(x) = \operatorname{len}\{\le y\} - \operatorname{len}\{\le x\}.$$

Hence $\operatorname{len}\{\le x\} + \operatorname{len}[x,y] \le \operatorname{len}\{\le y\}$.

(3) This follows immediately from the exactness of $\lambda_{\mathcal{L}}$.

(4) The function $\lambda_{\mathcal{K}} \circ \lambda\colon \mathcal{L} \to \mathbf{Ord}$ is strictly increasing, so, from 2.3(1), $\lambda_{\mathcal{L}}(x) \le \lambda_{\mathcal{K}}(\lambda(x))$ for all $x \in \mathcal{L}$. In particular,

$$\operatorname{len}\mathcal{L} = \lambda_{\mathcal{L}}(\top) \le \lambda_{\mathcal{K}}(\lambda(\top)) \le \lambda_{\mathcal{K}}(\top) = \operatorname{len}\mathcal{K}.$$

□

The identity map on **Ord** is strictly increasing and exact, so we have $\lambda_{\mathbf{Ord}}(\alpha) = \alpha$ for all ordinals α. In addition, $\operatorname{len}\{\le \alpha\} = \alpha$ and $\operatorname{len}[\alpha,\beta] = \beta - \alpha$ for all ordinals $\alpha \le \beta$, as can be easily checked.

Now consider $\mathbf{Ord} \times \mathbf{Ord}$. This partially ordered class is Artinian and for any $(\alpha,\beta) \in \mathbf{Ord} \times \mathbf{Ord}$ we have that $\{\le (\alpha,\beta)\} \cong \{\le \alpha\} \times \{\le \beta\}$ is a set. Thus $\mathbf{Ord} \times \mathbf{Ord}$ has a length function $\lambda_{\mathbf{Ord}\times\mathbf{Ord}}$ which we can use to define a new operation on ordinals:

Definition 2.7. *Define the* **natural sum** *of ordinals α and β by*

$$\alpha \oplus \beta = \lambda_{\mathbf{Ord}\times\mathbf{Ord}}(\alpha,\beta).$$

Note that, from 2.6(1), $\alpha \oplus \beta = \operatorname{len}\{\le (\alpha,\beta)\} = \operatorname{len}(\{\le \alpha\} \times \{\le \beta\})$.

The natural sum of ordinals was originally defined by G. Hessenberg [13, pages 591-594] as in Definition 2.11 (see also [10, page 363]). In 2.12, we will show that these two definitions for the natural sum are equivalent.

Lemma 2.8. *The operation $\oplus$ is commutative, associative and, for all $\alpha,\beta,\gamma \in \mathbf{Ord}$,*

(1) $\alpha \oplus \beta = \alpha \oplus \gamma \implies \beta = \gamma$

(2) $\alpha \oplus \beta \le \alpha \oplus \gamma \implies \beta \le \gamma$

Proof. Since the Cartesian product operation on partially ordered classes is commutative and associative, $\oplus$ is a commutative and associative operation on ordinals. The cancellation properties follow easily from the fact that, for a fixed $\alpha \in \mathbf{Ord}$, the map $\beta \mapsto (\alpha,\beta) \mapsto \lambda_{\mathbf{Ord}\times\mathbf{Ord}}(\alpha,\beta) = \alpha \oplus \beta$ is strictly increasing and hence injective. □

Notice that, being commutative, $\oplus$ is cancellative on both sides, unlike ordinary ordinal addition. Note also that for any $\alpha \in \mathbf{Ord}$ we have $0 \oplus \alpha = \alpha \oplus 0 = \operatorname{len}(\{\le \alpha\} \times \{0\}) = \operatorname{len}\{\le \alpha\} = \alpha$.

The importance of this operation is already apparent in the following easy theorem.

Theorem 2.9. *Let $\mathcal{K}$ and $\mathcal{L}$ be partially ordered classes with length functions. Then $\lambda_{\mathcal{K}\times\mathcal{L}}(x,y) = \lambda_{\mathcal{K}}(x) \oplus \lambda_{\mathcal{L}}(y)$ for all $(x,y) \in \mathcal{K}\times\mathcal{L}$. In particular,* $\operatorname{len}(\mathcal{K}\times\mathcal{L}) = \operatorname{len}\mathcal{K} \oplus \operatorname{len}\mathcal{L}$ *if $\mathcal{L}$ and $\mathcal{K}$ have maximum elements.*

Proof. The map $\lambda\colon \mathcal{K}\times\mathcal{L} \to \mathbf{Ord}$ defined by $\lambda(x,y) = \lambda_{\mathcal{K}}(x) \oplus \lambda_{\mathcal{L}}(y)$ is the composition of the strictly increasing exact functions $\lambda_{\mathcal{K}} \times \lambda_{\mathcal{L}}$ and $\lambda_{\mathbf{Ord}\times\mathbf{Ord}}$, and so is itself strictly increasing and exact. Hence, from 2.3, we get $\lambda = \lambda_{\mathcal{K}\times\mathcal{L}}$. □

It turns out that the natural sum of two ordinals can be calculated very easily from their normal forms. To show this we need the following lemma.

Lemma 2.10. *Let $\alpha,\beta,\alpha_1,\beta_1,\dots,\alpha_n,\beta_n \in \mathbf{Ord}$.*

(1) $\alpha+\beta \le \alpha\oplus\beta$
(2) $(\alpha_1\oplus\beta_1)+\dots+(\alpha_n\oplus\beta_n) \le (\alpha_1+\alpha_2+\dots+\alpha_n)\oplus(\beta_1+\beta_2+\dots+\beta_n)$
(3) $\alpha_1+\beta_1+\dots+\alpha_n+\beta_n \le (\alpha_1+\alpha_2+\dots+\alpha_n)\oplus(\beta_1+\beta_2+\dots+\beta_n)$

Proof. For claim 2, set $x_0 = (0,0)$, $x_1 = (\alpha_1,\beta_1)$, $x_2 = (\alpha_1+\alpha_2, \beta_1+\beta_2)$, $\dots$, $x_n = (\alpha_1+\alpha_2+\dots+\alpha_n, \beta_1+\beta_2+\dots+\beta_n)$ in $\mathbf{Ord}\times\mathbf{Ord}$. Then for $i = 1,2,\dots,n$ we have $[x_{i-1},x_i] \cong \{\le \alpha_i\} \times \{\le \beta_i\}$ so $\operatorname{len}[x_{i-1},x_i] = \alpha_i \oplus \beta_i$.

Applying 2.6(2) inductively to the sequence $x_0 \le x_1 \le \ldots \le x_n$ yields

$$\operatorname{len}[x_0,x_1] + \operatorname{len}[x_1,x_2] + \ldots + \operatorname{len}[x_{n-1},x_n] \le \operatorname{len}[x_0,x_n].$$

Rewriting this in terms of α_i and β_i yields 2.

Claim 1 is a special case of 2: $\alpha+\beta = (\alpha\oplus 0)+(0\oplus\beta) \le (\alpha+0)\oplus(0+\beta) = \alpha\oplus\beta$. Then, since $\alpha_i+\beta_i \le \alpha_i\oplus\beta_i$, for $i = 1,2,\dots,n$, the inequality 3 is immediate from 2. □

Consider now the natural sum of two ordinals which are given in short normal form, for example, $\alpha = \omega^\omega + \omega^3 + \omega 2 + 1$ and $\beta = \omega^3 + \omega$. Using 2.10(3), we can interleave the terms of these two normal forms in various ways and add them to get lower bounds for $\alpha\oplus\beta$. There is a unique way of doing this so that no terms are lost, namely: Write down the terms gathered from both the short normal forms in decreasing order and then add. In the example, we have the six terms $\omega^\omega, \omega^3, \omega^3, \omega 2, \omega, 1$ so, from the lemma,

$$\omega^\omega + \omega^3 + \omega^3 + \omega 2 + \omega + 1 = \omega^\omega + \omega^3 2 + \omega 3 + 1 \le \alpha\oplus\beta.$$

We will show that this method actually gives us the natural sum of α and β, not just a lower bound for it, but first we need to formalize this construction:

Definition 2.11. *Let α and β be nonzero ordinals. With suitable relabeling, the short normal forms for these ordinals can be written using the same strictly decreasing set of exponents $\gamma_1 > \gamma_2 > \cdots > \gamma_n$:*

$$\alpha = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n \qquad \beta = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$$

where $n_i, m_i \in \mathbb{Z}^+$, that is, we allow m_i, n_i to be zero.

Now we define the operation $\oplus'$ by

$$\alpha \oplus' \beta = \omega^{\gamma_1}(m_1 + n_1) + \omega^{\gamma_2}(m_2 + n_2) + \cdots + \omega^{\gamma_n}(m_n + n_n).$$

This is a well defined operation because of the uniqueness of the normal forms for ordinals. In addition, we define $0 \oplus' \alpha = \alpha \oplus' 0 = \alpha$, and $0 \oplus' 0 = 0$.

Theorem 2.12. *The operations $\oplus$ and $\oplus'$ are identical.*

Proof. From the discussion following 2.10 we have $\alpha \oplus' \beta \leq \alpha \oplus \beta$ for any ordinals α and β. To show the opposite inequality we need only show that $\oplus'$: $\mathbf{Ord} \times \mathbf{Ord} \to \mathbf{Ord}$ is strictly increasing:

Since $\oplus'$ is commutative and increasing, it suffices to show only that, for all $\alpha, \beta \in \mathbf{Ord}$, we have $\alpha \oplus' (\beta + 1) > \alpha \oplus' \beta$. But this follows easily from the definition of $\oplus'$, in fact, $\alpha \oplus' (\beta + 1) = (\alpha \oplus' \beta) + 1$. □

It is apparent from this theorem and 2.11 that, if α and β are finite ordinals, then $\alpha \oplus \beta = \alpha + \beta$ and hence both ordinal addition and $\oplus$ coincide with the usual addition of integers.

Notice also that $\operatorname{Krank}(\alpha \oplus \beta) = \operatorname{Krank} \alpha + \operatorname{Krank} \beta$ for any ordinals $\alpha, \beta \in \mathbf{Ord}$, so that, if $\mathcal{K}$ and $\mathcal{L}$ are partially ordered classes with length functions and maximum elements, then $\operatorname{Krank}(\mathcal{L} \times \mathcal{K}) = \operatorname{Krank} \mathcal{L} + \operatorname{Krank} \mathcal{K}$.

3. The Length of a Bounded Artinian Modular Lattice

We will now specialize to the case of length functions on bounded Artinian modular lattices. In this section we will no longer need to consider proper classes, and so we define a **lattice** to be a partially ordered set $\mathcal{L}$ such that every pair of elements, $x, y \in \mathcal{L}$, has a supremum, $x \vee y$, and an infimum, $x \wedge y$. A **bounded lattice** is a lattice which has a maximum element $\top$ and a minimum element $\bot$. A lattice $\mathcal{L}$ is **modular** if $(x_1 \leq x_2 \implies (x_1 \vee y) \wedge x_2 = x_1 \vee (y \wedge x_2)\ )$ for all $x_1, x_2, y \in \mathcal{L}$.

Lemma 3.1. *Let $\mathcal{L}$ be a modular lattice and $x, y \in \mathcal{L}$.*

(1) *The map $\lambda\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L} \times \mathcal{L}$ given by $(x, y) \mapsto (x \wedge y, x \vee y)$, is strictly increasing.*
(2) *The maps $\phi\colon [x, y \vee x] \to [y \wedge x, y]$ given by $z \mapsto z \wedge y$, and $\psi\colon [y \wedge x, y] \to [x, y \vee x]$ given by $z \mapsto z \vee x$ are inverse isomorphisms.*

Proof. (1) Suppose that $(x_1, y_1) \leq (x_2, y_2)$ with $\lambda(x_1, y_1) = \lambda(x_2, y_2)$. Then $x_1 \leq x_2$, $y_1 \leq y_2$, $x_1 \wedge y_1 = x_2 \wedge y_2$ and $x_1 \vee y_1 = x_2 \vee y_2$. Using modularity we get

$$x_2 = (x_2 \vee y_2) \wedge x_2 = (x_1 \vee y_1) \wedge x_2 = x_1 \vee (y_1 \wedge x_2)$$
$$\leq x_1 \vee (y_2 \wedge x_2) = x_1 \vee (y_1 \wedge x_1) = x_1.$$

Hence $x_1 = x_2$, and by symmetry, $y_1 = y_2$. Thus $(x_1, y_1) = (x_2, y_2)$.

Now suppose $(x_1, y_1) < (x_2, y_2)$. Since λ is an increasing function, we have $\lambda(x_1, y_1) \leq \lambda(x_2, y_2)$. From the above argument, $\lambda(x_1, y_1) = \lambda(x_2, y_2)$ is impossible, and so we must have $\lambda(x_1, y_1) < \lambda(x_2, y_2)$.

(2) [14, Theorem 13, page 13] The functions ψ and ϕ are clearly increasing.

If $z \in [x, y \vee x]$, then $\psi(\phi(z)) = (z \wedge y) \vee x = (y \vee x) \wedge z = z$. The second equality comes from applying the modularity of $\mathcal{L}$ to the inequality $x \leq z$. Thus $\psi \circ \phi$ is the identity map on $[x, y \vee x]$, and similarly $\phi \circ \psi$ is the identity map on $[y \wedge x, y]$. □

From 2.5 we know that a bounded Artinian modular lattice $\mathcal{L}$ has a length function. The main property of the length function in this circumstance is that if $x \in \mathcal{L}$ then $\operatorname{len}[\bot, x] \oplus \operatorname{len}[x, \top]$ is an upper bound for $\operatorname{len} \mathcal{L}$. Without the hypothesis that $\mathcal{L}$ is a modular lattice, we know only a lower bound, namely $\operatorname{len}[\bot, x] + \operatorname{len}[x, \top]$.

Theorem 3.2. *Let x and y be elements of a bounded Artinian modular lattice $\mathcal{L}$.*

(1) $\operatorname{len}[\bot, x] + \operatorname{len}[x, \top] \leq \operatorname{len} \mathcal{L} \leq \operatorname{len}[\bot, x] \oplus \operatorname{len}[x, \top]$
(2) $\operatorname{len}[\bot, x \wedge y] + \operatorname{len}[\bot, x \vee y] \leq \operatorname{len}[\bot, x] \oplus \operatorname{len}[\bot, y]$
$\leq \operatorname{len}[\bot, x \wedge y] \oplus \operatorname{len}[\bot, x \vee y]$
(3) $\operatorname{len}[x \wedge y, \top] + \operatorname{len}[x \vee y, \top] \leq \operatorname{len}[x, \top] \oplus \operatorname{len}[y, \top]$
$\leq \operatorname{len}[x \wedge y, \top] \oplus \operatorname{len}[x \vee y, \top]$

Proof. (1) The first inequality is directly from 2.6(2). To prove the second inequality, consider the restriction of the map λ from 3.1(1) to the domain $\mathcal{L} \times \{x\}$. This map is strictly increasing and its image is contained in $[\bot, x] \times [x, \top]$. From 2.6(4) we get

$$\operatorname{len} \mathcal{L} = \operatorname{len}(\mathcal{L} \times \{x\}) \leq \operatorname{len}([\bot, x] \times [x, \top]) = \operatorname{len}[\bot, x] \oplus \operatorname{len}[x, \top].$$

(2) To prove the first inequality we apply 1 to the lattices $[\bot, x \vee y]$ and $[\bot, y]$. This yields $\operatorname{len}[\bot, x \vee y] \leq \operatorname{len}[\bot, x] \oplus \operatorname{len}[x, x \vee y]$ and $\operatorname{len}[\bot, x \wedge y] + \operatorname{len}[x \wedge y, y] \leq \operatorname{len}[\bot, y]$ respectively. From 3.1(2) we also have $\operatorname{len}[x, x \vee y] = \operatorname{len}[x \wedge y, y]$. Hence

$$\begin{aligned} \operatorname{len}[\bot, x \wedge y] + \operatorname{len}[\bot, x \vee y] &\leq \operatorname{len}[\bot, x \wedge y] + (\operatorname{len}[\bot, x] \oplus \operatorname{len}[x, x \vee y]) \\ &= \operatorname{len}[\bot, x \wedge y] + (\operatorname{len}[\bot, x] \oplus \operatorname{len}[x \wedge y, y]) \\ &\leq \operatorname{len}[\bot, x] \oplus (\operatorname{len}[\bot, x \wedge y] + \operatorname{len}[x \wedge y, y]) \\ &\leq \operatorname{len}[\bot, x] \oplus \operatorname{len}[\bot, y] \end{aligned}$$

We have also used the fact that $\alpha_1 + (\alpha_2 \oplus \beta_2) \leq (\alpha_1 + \alpha_2) \oplus \beta_2$ which follows from 2.10(2).

To prove the second inequality, consider the restriction of the map λ from 3.1(1) to the domain $[\bot, x] \times [\bot, y]$. This map is strictly increasing and its image is contained in $[\bot, x \wedge y] \times [\bot, x \vee y]$, and so, from 2.6(4) we get $\operatorname{len}[\bot, x] \oplus \operatorname{len}[\bot, y] \leq \operatorname{len}[\bot, x \wedge y] \oplus \operatorname{len}[\bot, x \vee y]$.

(3) Proof is similar to that of 2. □

Note that, if $\operatorname{len} \mathcal{L}$ is finite, then ordinal addition and $\oplus$ coincide and all the inequalities in this theorem become equalities.

One important corollary of this theorem follows from the observation that for any nonzero ordinals α and β, $\alpha + \beta$ and $\alpha \oplus \beta$ have the same leading term in their short normal forms. Further, this leading term depends only on the leading terms of α and β. For example, if $\alpha = \omega^\omega + \omega^3 + \omega 2 + 1$ and $\beta = \omega^3 + \omega$, then $\alpha + \beta = \omega^\omega + \omega^3 2 + \omega$ and $\alpha \oplus \beta = \omega^\omega + \omega^3 2 + \omega 3 + 1$, both having the leading term ω^ω.

Formulating this observation in terms of Krull dimension and γ-length we have, for example, from 3.2(1)

Lemma 3.3. *Let $\mathcal{L}$ be a bounded Artinian modular lattice, $\gamma = \operatorname{Kdim} \mathcal{L}$ and $x \in \mathcal{L}$.*

(1) $\operatorname{Kdim} \mathcal{L} = \max\{\operatorname{Kdim}[\bot, x], \operatorname{Kdim}[x, \top]\}$
(2) $\operatorname{len}_\gamma \mathcal{L} = \operatorname{len}_\gamma[\bot, x] + \operatorname{len}_\gamma[x, \top]$

We leave it to the reader to write down the corresponding lemma derived from 3.2(2 and 3).

Theorem 3.2 directs our attention to pairs of ordinals α and β such that $\alpha + \beta = \alpha \oplus \beta$. Some easy ordinal arithmetic shows when this happens:

Lemma 3.4. *Suppose* $\alpha + \beta = \alpha \oplus \beta = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ *in long normal form. Then* $\alpha = 0$ *or* $\beta = 0$*, or there is some* $i \in \{1, 2, \ldots, n-1\}$ *such that* $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_i}$ *and* $\beta = \omega^{\gamma_{i+1}} + \omega^{\gamma_{i+2}} + \cdots + \omega^{\gamma_n}$*.*

Lemma 3.5. *Let* $\mathcal{L}$ *be a bounded Artinian modular lattice. Suppose we have* $\alpha, \beta \in \mathbf{Ord}$ *such that* $\alpha + \beta = \alpha \oplus \beta$*.*

(1) *If* $\operatorname{len} \mathcal{L} = \alpha + \beta$*, then there is some* $x \in \mathcal{L}$ *such that* $\operatorname{len}[\bot, x] = \alpha$ *and* $\operatorname{len}[x, \top] = \beta$*.*
(2) *If there is* $x \in \mathcal{L}$ *such that* $\operatorname{len}[\bot, x] = \alpha$ *and* $\operatorname{len}[x, \top] = \beta$*, then* $\operatorname{len} \mathcal{L} = \alpha + \beta$*.*

Proof. (1) From 2.6(3), there is some $x \in \mathcal{L}$ such that $\operatorname{len}[\bot, x] = \alpha$. From 3.2(1), $\alpha + \operatorname{len}[x, \top] \leq \alpha + \beta = \alpha \oplus \beta \leq \alpha \oplus \operatorname{len}[x, \top]$. Cancellation in the first inequality gives $\operatorname{len}[x, \top] \leq \beta$. Cancellation in the second inequality gives $\beta \leq \operatorname{len}[x, \top]$. Hence $\operatorname{len}[x, \top] = \beta$
(2) This follows directly from 3.2(1). □

For example, if $\operatorname{len} \mathcal{L} = \omega^\omega + \omega^3 2 + 1$, then the previous two lemmas guarantee the existence of an $x \in \mathcal{L}$ such that $\operatorname{len}[x, \top]$ is any of the following ordinals:

$$0,\ 1,\ \omega^3 + 1,\ \omega^3 2 + 1,\ \omega^\omega + \omega^3 2 + 1.$$

Lemma 3.6. *Let* $\mathcal{L}$ *be a bounded Artinian modular lattice with* $\operatorname{len} \mathcal{L} \neq 0$ *(that is,* $\mathcal{L}$ *is nontrivial). Then the following are equivalent:*

(1) $\operatorname{len} \mathcal{L} = \omega^\gamma$ *for some* $\gamma \in \mathbf{Ord}$
(2) $\operatorname{len}[x, \top] = \operatorname{len} \mathcal{L}$ *for all* $x < \top$
(3) $\operatorname{Kdim}[\bot, x] < \operatorname{Kdim} \mathcal{L}$ *for all* $x < \top$

Proof. Put $\alpha = \operatorname{len} \mathcal{L}$. Since, by 2.6(3), for any $\beta < \alpha$ there is some x in $\mathcal{L}$ such that $\operatorname{len}[\bot, x] = \beta$, the claim is an easy consequence of 3.2(1), 3.3 and 2.2(2). □

Definition 3.7. *A nontrivial bounded Artinian modular lattice* $\mathcal{L}$ *is* **critical** *if it satisfies any of the conditions of the previous lemma. More specifically, we will say* $\mathcal{L}$ *is* γ**-critical** *if* $\operatorname{len} \mathcal{L} = \omega^\gamma$*.*

A **critical series** *for a bounded Artinian modular lattice* $\mathcal{L}$*, is a sequence*

$$\bot = z_0 < z_1 < \cdots < z_n = \top$$

in $\mathcal{L}$ *such that* $[z_{i-1}, z_i]$ *is* γ_i*-critical for all* i*, and* $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$*.*

If $\operatorname{len} \mathcal{L} = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form, then from 3.4 and 3.5, there is an element $z \in \mathcal{L}$ such that $\operatorname{len}[z, \top] = \omega^{\gamma_n}$ and $\operatorname{len}[\bot, z] = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_{n-1}}$. In particular, $[z, \top]$ is γ_n-critical. A simple induction then shows that any nontrivial bounded Artinian modular lattice has a critical series:

Lemma 3.8. *Let $\mathcal{L}$ be a bounded Artinian modular lattice. Then the following are equivalent*

(1) $\operatorname{len}\mathcal{L} = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ *in long normal form.*
(2) $\mathcal{L}$ *has a critical series* $\bot = z_0 < z_1 < \cdots < z_n = \top$ *with* $[z_{i-1}, z_i]$ γ_i*-critical for* $i = 1, 2, \ldots, n$.

Given a bounded Artinian modular lattice $\mathcal{L}$ with a critical series, the next theorem shows how to construct a critical series for the sublattice $[x, \top]$ for any element $x \in \mathcal{L}$.

Theorem 3.9. *Let $\mathcal{L}$ be a bounded Artinian modular lattice with the critical series* $\bot = z_0 < z_1 < \cdots < z_n = \top$ *with* $[z_{i-1}, z_i]$ γ_i*-critical for* $i = 1, 2, \ldots, n$. *Let* $x \in \mathcal{L}$ *and set* $x_i = x \vee z_i$ *for* $i = 0, 1, 2, \ldots, n$. *Then for* $i = 1, 2, \ldots, n$, $\operatorname{len}[x_{i-1}, x_i]$ *is either zero or* ω^{γ_i}. *Further, the sequence* $x = x_0 \leq x_2 \leq \cdots \leq x_n = \top$, *after removal of duplicate entries, is a critical series for* $[x, \top]$.

Proof. From 3.1(2) we get $[x_{i-1}, z_i \vee x_{i-1}] \cong [z_i \wedge x_{i-1}, z_i]$ for $i = 1, 2, \ldots, n$. Since $z_i \vee x_{i-1} = z_i \vee (z_{i-1} \vee x) = z_i \vee x = x_i$, and, using the modularity of the lattice, $z_i \wedge x_{i-1} = z_i \wedge (z_{i-1} \vee x) = z_{i-1} \vee (z_i \wedge x)$, we get $[x_{i-1}, x_i] \cong [z_{i-1} \vee (z_i \wedge x), z_i]$. We also have $z_{i-1} \leq z_{i-1} \vee (z_i \wedge x) \leq z_i$, and so $[x_{i-1}, x_i]$ is isomorphic to a final segment of $[z_{i-1}, z_i]$. Because $[z_{i-1}, z_i]$ is γ_i-critical, 3.6(2) applies and either $x_{i-1} = x_i$ or $\operatorname{len}[x_{i-1}, x_i] = \omega^{\gamma_i}$.

The claim that $x = x_0 \leq x_2 \leq \cdots \leq x_n = \top$, after removal of duplicate entries, is a critical series for $[x, \top]$ is then clear. □

From this theorem we see that the factors in a critical series for $[x, \top]$ have lengths which are among the lengths of the factors in a critical series for $\mathcal{L}$. Combining this with 3.8 we get

Corollary 3.10. *Let $\mathcal{L}$ be a bounded Artinian modular lattice with* $\operatorname{len}\mathcal{L} = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$ *in short normal form. Then for* $x \in \mathcal{L}$,

$$\operatorname{len}[x, \top] = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n$$

for some $m_i \in \mathbb{Z}^+$ *such that* $m_i \leq n_i$ *for all* i. *In particular,*

(1) $\operatorname{Krank}[x, \top] \leq \operatorname{Krank}\mathcal{L}$ *with equality if and only if* $\operatorname{len}[x, \top] = \operatorname{len}\mathcal{L}$.
(2) $\operatorname{Kdim}[x, \top] \in \{-1, \gamma_1, \gamma_2, \ldots, \gamma_n\}$.

Continuing with the example $\operatorname{len}\mathcal{L} = \omega^\omega + \omega^3 2 + 1$, if $x \in \mathcal{L}$, then $\operatorname{len}[x, \top]$ is one of the following ordinals:

$$0,\ 1,\ \omega^3,\ \omega^3 + 1,\ \omega^3 2,\ \omega^3 2 + 1,\ \omega^\omega,\ \omega^\omega + 1,$$
$$\omega^\omega + \omega^3,\ \omega^\omega + \omega^3 + 1,\ \omega^\omega + \omega^3 2,\ \omega^\omega + \omega^3 2 + 1$$

This result is to be contrasted with the possible values of $\operatorname{len}[\perp, x]$ which, by 2.6(3), include all ordinals less than $\omega^\omega + \omega^3 2 + 1$.

The mere fact that $\operatorname{len}[x, \top]$ can take on only a finite number of different values is significant. As an application of this we prove a simple property of complemented lattices: A bounded lattice $\mathcal{L}$ is **complemented** if for every $x \in \mathcal{L}$ there is some $y \in \mathcal{L}$, called a **complement** of x, such that $x \vee y = \top$ and $x \wedge y = \perp$.

Corollary 3.11. [15, 0.4] *Any complemented Artinian modular lattice has finite length.*

Proof. Let $\mathcal{L}$ be such a lattice. Then for any $\alpha \leq \operatorname{len} \mathcal{L}$, there is some $x \in \mathcal{L}$ such that $\operatorname{len}[\perp, x] = \alpha$. From 3.1(2) we have $\operatorname{len}[\perp, x] = \operatorname{len}[y, \top]$ where y is a complement of x. But by 3.10, there are only a finite number of possible values for $\operatorname{len}[y, \top]$. Thus $\operatorname{len} \mathcal{L}$ is also finite. □

4. The Length of a Noetherian Module

In this section we apply the results of our study of length functions to Noetherian modules. Throughout this section, R will be a fixed ring and R-**Noeth** the category of Noetherian left R-modules. If $A \in R$-**Noeth**, we will write $\mathcal{L}(A)$ for the set of submodules of A ordered by set inclusion, and $\mathcal{L}^\circ(A)$ for the set of submodules of A ordered by reverse set inclusion, that is, the dual of $\mathcal{L}(A)$. Since A is Noetherian, $\mathcal{L}^\circ(A)$ is Artinian. In addition, both $\mathcal{L}(A)$ and $\mathcal{L}^\circ(A)$ are bounded modular lattices. In particular, in $\mathcal{L}^\circ(A)$, we have $A_1 \wedge A_2 = A_1 + A_2$ and $A_1 \vee A_2 = A_1 \cap A_2$ for all A_1, A_2 in $\mathcal{L}^\circ(A)$. For the details of these claims about $\mathcal{L}(A)$, see L. Rowen, *Ring Theory, Volume 1*, [16, pages 7-9].

Since $\mathcal{L}^\circ(A)$ is a bounded Artinian partially ordered set, we can define the **length**, **Krull dimension**, **Krull rank** and **γ-length** of A by

$$\operatorname{len} A = \operatorname{len} \mathcal{L}^\circ(A) \qquad \operatorname{Kdim} A = \operatorname{Kdim} \mathcal{L}^\circ(A)$$
$$\operatorname{Krank} A = \operatorname{Krank} \mathcal{L}^\circ(A) \qquad \operatorname{len}_\gamma A = \operatorname{len}_\gamma \mathcal{L}^\circ(A).$$

If A is a finite length module, then $\operatorname{len} A$ is finite and has the usual meaning as the length of a composition series for A. We will show in 4.10 that the definition of Krull dimension here coincides with the usual one for Noetherian modules.

Let $0 \to A \to B \to C \to 0$ be an exact sequence in R-**Noeth**, and A' the image of A in B. Then $\mathcal{L}^\circ(C) \cong [B, A'] \subseteq \mathcal{L}^\circ(B)$, and $\mathcal{L}^\circ(A) \cong \mathcal{L}^\circ(A') \cong [A', 0] \subseteq \mathcal{L}^\circ(B)$. So, from 3.2(1) and 3.3, we get the main result of this section:

Theorem 4.1. [1, 2.1] *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence in* R-**Noeth** *and* $\gamma = \operatorname{Kdim} B$.

(1) $\operatorname{len} C + \operatorname{len} A \leq \operatorname{len} B \leq \operatorname{len} C \oplus \operatorname{len} A$
(2) $\operatorname{Kdim} B = \max\{\operatorname{Kdim} A, \operatorname{Kdim} C\}$
(3) $\operatorname{len}_\gamma B = \operatorname{len}_\gamma A + \operatorname{len}_\gamma C$

Of course, if A, B, C are finite length modules, then $\operatorname{len} A$, $\operatorname{len} B$ and $\operatorname{len} C$ are finite ordinals; ordinal addition, $\oplus$ and the usual addition of natural numbers coincide; and the inequalities in this theorem become equalities.

Corollary 4.2. *If* $A, B \in R$-**Noeth** *and* $\phi\colon A \to B$*, then* ϕ *is injective if and only if* $\operatorname{len} A = \operatorname{len} \phi(A)$.

Proof. We have the short exact sequence $0 \to \ker\phi \to A \to \phi(A) \to 0$. So from 4.1, $\operatorname{len}\phi(A) + \operatorname{len}(\ker\phi) \leq \operatorname{len} A$. If $\operatorname{len} A = \operatorname{len}\phi(A)$, then we can cancel from this inequality to get $\operatorname{len}(\ker\phi) = 0$ and hence $\ker\phi = 0$. The converse implication is clear since if ϕ is injective, then $\phi(A) \cong A$. □

A simple special case of this corollary is the well known property of $A \in R$-**Noeth** that if $\phi \in \operatorname{End} A$ is surjective, then ϕ is injective.

From 3.2(2,3) we get the following.

Lemma 4.3. *Let* $A_1, A_2 \leq A \in R$-**Noeth**.

(1) $\operatorname{len}(A/(A_1 + A_2)) + \operatorname{len}(A/(A_1 \cap A_2)) \leq \operatorname{len}(A/A_1) \oplus \operatorname{len}(A/A_2)$
$\leq \operatorname{len}(A/(A_1 + A_2)) \oplus \operatorname{len}(A/(A_1 \cap A_2))$
(2) $\operatorname{Kdim}(A/(A_1 \cap A_2)) = \max\{\operatorname{Kdim}(A/A_1), \operatorname{Kdim}(A/A_2)\}$
(3) $\operatorname{len}(A_1 + A_2) + \operatorname{len}(A_1 \cap A_2) \leq \operatorname{len} A_1 \oplus \operatorname{len} A_2$
$\leq \operatorname{len}(A_1 + A_2) \oplus \operatorname{len}(A_1 \cap A_2)$
(4) $\operatorname{Kdim}(A_1 + A_2) = \max\{\operatorname{Kdim} A_1, \operatorname{Kdim} A_2\}$

Claim 3 of this lemma can also be obtained by applying 4.1(1) to the usual exact sequence $0 \to A_1 \cap A_2 \to A_1 \oplus A_2 \to A_1 + A_2 \to 0$. Claims 2 and 4 are easy consequences of 1 and 3 obtained as in 3.3.

Lemma 4.4. *Let* $A, B, C \in R$-**Noeth** *and* $n \in \mathbb{N}$.

(1) $\operatorname{len}(A \oplus B) = \operatorname{len} A \oplus \operatorname{len} B$
(2) $A \oplus C \cong B \oplus C \implies \operatorname{len} A = \operatorname{len} B$
(3) $A^n \cong B^n \implies \operatorname{len} A = \operatorname{len} B$

Proof. (1) The special case of 4.3(3) when $A_1 \cap A_2 = 0$.
(2) From $A \oplus C \cong B \oplus C$ and 1 we have $\operatorname{len} A \oplus \operatorname{len} C = \operatorname{len} B \oplus \operatorname{len} C$. Since $\oplus$ is cancellative (2.8) it follows that $\operatorname{len} A = \operatorname{len} B$.
(3) Follows, as for 2, from the cancellativity of $\oplus$. □

We will write $\operatorname{udim} A$ for the uniform (or Goldie) dimension of a module A [17, 2.2.10]. Krull rank has many properties in common with uniform dimension.

Lemma 4.5. *Let $A, B \in R$-**Noeth**.*

(1) *If $A \leq B$, then $\operatorname{Krank} A \leq \operatorname{Krank} B$ with equality if and only if $\operatorname{len} A = \operatorname{len} B$.*
(2) $\operatorname{Krank}(A \oplus B) = \operatorname{Krank} A + \operatorname{Krank} B$
(3) $\operatorname{udim} A \leq \operatorname{Krank} A$

Proof. (1) Immediate from 3.10.
(2) From 4.4(1) and the fact that $\operatorname{Krank}(\alpha \oplus \beta) = \operatorname{Krank} \alpha + \operatorname{Krank} \beta$ for any ordinals $\alpha, \beta \in \mathbf{Ord}$.
(3) Any nonzero module has nonzero Krull rank, so if A contains a direct sum of $\operatorname{udim} A$ nonzero submodules, then using 1 and 2, we must have $\operatorname{udim} A \leq \operatorname{Krank} A$. □

Applying 2.6(3), 3.4, 3.5 and 3.10 to Noetherian modules we get the following.

Theorem 4.6. *Let $A \in R$-**Noeth**.*

(1) *For every ordinal $\alpha \leq \operatorname{len} A$ there exists a submodule $A' \leq A$ such that $\operatorname{len}(A/A') = \alpha$.*
(2) *Suppose $\operatorname{len} A = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form. Then for ordinals $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_i}$ and $\beta = \omega^{\gamma_{i+1}} + \omega^{\gamma_{i+2}} + \cdots + \omega^{\gamma_n}$, for some $i \in \{0, 1, 2, \ldots, n-1\}$ there exists a submodule $A' \leq A$ such that $\operatorname{len}(A/A') = \alpha$ and $\operatorname{len} A' = \beta$.*
(3) *Suppose $\operatorname{len} A = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$ in short normal form. Then for any submodule $A' \leq A$ we have $\operatorname{len} A' = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n$ for some $m_i \in \mathbb{Z}^+$ such that $m_i \leq n_i$ for all i. In particular, $\operatorname{Kdim} A' \in \{-1, \gamma_1, \gamma_2, \ldots, \gamma_n\}$.*

For example, suppose that $\operatorname{len} A = \omega^\omega + \omega^3 2 + 1$. Then 4.6(2) guarantees the existence of submodules of A with lengths

$$0,\ 1,\ \omega^3 + 1,\ \omega^3 2 + 1,\ \omega^\omega + \omega^3 2 + 1,$$

and 4.6(3) says that the length of any submodule of A is one of the following ordinals:

$$\begin{gathered} 0,\ \ 1,\ \ \omega^3,\ \ \omega^3 + 1,\ \ \omega^3 2,\ \ \omega^3 2 + 1,\ \omega^\omega,\ \ \omega^\omega + 1, \\ \omega^\omega + \omega^3,\ \ \omega^\omega + \omega^3 + 1,\ \ \omega^\omega + \omega^3 2,\ \ \omega^\omega + \omega^3 2 + 1 \end{gathered}$$

As we have seen already in 3.11, the fact that there are only a finite number of possible values for $\operatorname{len} A'$ when $A' \leq A$ is already useful.

Corollary 4.7. *Let* $A, B \in R$-**Noeth**. *Then the partially ordered set* $\mathcal{K} = \{\ker\phi \mid \phi : A \to B\} \subseteq \mathcal{L}^\circ(A)$ *has finite length.*

Proof. Consider the restriction of $\lambda_{\mathcal{L}^\circ(A)}$ to $\mathcal{K}$. This map is, of course, strictly increasing, so any chain in $\mathcal{K}$ maps injectively into $\lambda_{\mathcal{L}^\circ(A)}(\mathcal{K})$.

Given a homomorphism $\phi : A \to B$, we have $A/\ker\phi \cong \operatorname{im}\phi \leq B$ and so $\lambda_{\mathcal{L}^\circ(A)}(\ker\phi) = \operatorname{len}(A/\ker\phi) = \operatorname{len}(\operatorname{im}\phi)$. Thus $\lambda_{\mathcal{L}^\circ(A)}(\mathcal{K})$ is contained in the set of the lengths of all submodules of B. By 4.6(3), this set is finite, so there is a finite bound on the length of chains in $\lambda_{\mathcal{L}^\circ(A)}(\mathcal{K})$. This same bound then limits the length of chains in $\mathcal{K}$. □

As a special case of this corollary we have that, if R is a left Noetherian ring and $B \in R$-**Noeth**, then the set of annihilators of elements of B has finite length.

Finally in this section we need to show that, for Noetherian modules, Krull dimension as we have defined it coincides with the usual definition.

Definition 4.8. *The* **Krull dimension (in the sense of Gordon and Robson)** [4], [18, Chapter 13], *of a module A, which we will denote by* $\operatorname{Kdim}' A \in \{-1\} \cup \mathbf{Ord}$, *is defined inductively as follows:*

- $\operatorname{Kdim}' A = -1$ *if and only if* $A = 0$.
- *Let* $\gamma \in \mathbf{Ord}$ *and assume that we have defined which modules have* Kdim' *equal to* δ *for every* $\delta < \gamma$. *Then* $\operatorname{Kdim}' A = \gamma$ *if and only if*
  - (a) *A does not have* Kdim' *less than* γ, *and*
  - (b) *for every countable decreasing chain* $A_1 \geq A_2 \geq \dots$ *of submodules of A,* $\operatorname{Kdim}'(A_i/A_{i+1}) < \gamma$ *for all but finitely many indices.*

This definition does not provide a Kdim' for all modules. However, any Noetherian module has a Kdim'. See [18, 13.3].

Lemma 4.9. *Let* $A \in R$-**Noeth** *with* $\operatorname{Kdim} A = \gamma \neq -1$.

(1) *For any ordinal* $\delta < \gamma$, *there is an infinite sequence* $A_1 \geq A_2 \geq \dots$ *of submodules of A such that* $\operatorname{Kdim}(A_{i-1}/A_i) = \delta$ *for all i.*

(2) *If* $A_1 \geq A_2 \geq \dots$ *is an infinite sequence of submodules of A, then* $\operatorname{Kdim}(A_i/A_{i+1}) < \gamma$ *for all but a finite number of indices.*

Proof. (1) Set $A_1 = A$. Since $\operatorname{Kdim} A_1 = \gamma$, we have $\operatorname{len} A_1 \geq \omega^\gamma > \omega^\delta$. By 4.6(1), there is some $A_2 \leq A_1$ such that $\operatorname{len}(A_1/A_2) = \omega^\delta$. We have $\operatorname{Kdim}(A_1/A_2) = \delta$ and $\operatorname{Kdim} A_1 = \max\{\operatorname{Kdim} A_2, \operatorname{Kdim}(A_1/A_2)\}$, so $\operatorname{Kdim} A_2 = \gamma$, and we can repeat the process to get $A_3, A_4 \dots$ as required.

(2) Since the sequence of ordinals $\operatorname{len} A_1 \geq \operatorname{len} A_2 \geq \dots$ is decreasing, there is some $n \in \mathbb{N}$ such that $\operatorname{len} A_{i+1} = \operatorname{len} A_n$ for all $i \geq n$. If $i \geq$

n, then, from 4.1(1), $\operatorname{len}(A_i/A_{i+1}) + \operatorname{len} A_{i+1} \leq \operatorname{len} A_i = \operatorname{len} A_{i+1}$, and hence, from 2.2(1), $\operatorname{Kdim}(A_i/A_{i+1}) < \operatorname{Kdim} A_{i+1} \leq \operatorname{Kdim} A = \gamma$. □

Theorem 4.10. *For all $A \in R$-**Noeth**,* $\operatorname{Kdim} A = \operatorname{Kdim}' A$.

Proof. Suppose the claim is not true. Let A be a counterexample of smallest possible Kdim. Set $\gamma = \operatorname{Kdim} A$. Then $\gamma > -1$ and for any module B with $\operatorname{Kdim} B < \gamma$ we have $\operatorname{Kdim} B = \operatorname{Kdim}' B$.

From 4.9(1), we see that, for any $\delta < \gamma$, the module A fails part (b) of the definition of having Kdim′ equal to δ. Thus A does not have Kdim′ less than γ, and A satisfies part (a) of the definition of having $\operatorname{Kdim}' A = \gamma$.

Also, by 4.9(2), A satisfies part (b) of this definition. Thus $\operatorname{Kdim}' A = \gamma$, and A is not a counterexample. □

Let $\gamma \in$ **Ord**. Then a module $A \in R$-**Noeth** is **γ-critical** [18, page 227] if $\operatorname{Kdim} A = \gamma$ and $\operatorname{Kdim}(A/A') < \gamma$ for all nonzero submodules $A' \leq A$. From 3.6 and 3.7 is is clear that a Noetherian module A is γ-critical if and only if $\mathcal{L}^\circ(A)$ is γ-critical if and only if $\operatorname{len} A = \omega^\gamma$. More generally $A \neq 0$ is critical if and only if $\operatorname{len} A' = \operatorname{len} A$ for all nonzero submodules $A' \leq A$.

Notice in particular that from 4.6(2), any nonzero Noetherian module contains a critical submodule. Specifically, if $\operatorname{len} A = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form, then A has a submodule of length ω^{γ_n}.

A **critical series** [18, page 229] for a module A is a submodule series $0 = A_n < A_{n-1} < \ldots < A_0 = A$ such that A_{i-1}/A_i is a γ_i-critical module for $i = 1, 2, \ldots, n$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$. Comparison with 3.7, shows that a submodule series $0 = A_n < A_{n-1} < \ldots < A_0 = A$ is a critical series if and only if $\perp = A = A_0 < A_1 < \ldots < A_n = 0 = \top$ is a critical series in $\mathcal{L}^\circ(A)$.

From 3.8 we see that an equivalent definition of the length of a nonzero Noetherian module A is $\operatorname{len} A = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ where $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ are the Krull dimensions of the factors in a critical series for A. Of course, for this to be a useful definition, it is necessary to establish first the existence of critical series and then the uniqueness of the Krull dimensions of the factors of such series. See [18, 13.9].

5. Some Applications

In this section we demonstrate the use of length in proving some familiar properties of left Noetherian rings. The important thing to notice in this section is that no further use is made of the ascending chain condition. The main theorems are proved by finite induction on the Krull rank of some module involved. The lemmas require only ordinal arithmetic, 4.1 and 4.4 in their proofs.

Many of the results in this section can be seen as providing conditions under which an ideal or module contains a big cyclic submodule. Here "big" has the following technical meaning:

Definition 5.1. *Given a module $A \in R$-**Noeth**, any submodule $A' \le A$ such that $\operatorname{len} A' = \operatorname{len} A$ is said to be* **big** *in A. This situation is denoted $A' \trianglelefteq A$.*

Of course, if A is a finite length module and $A' \trianglelefteq A$, then $A' = A$. Other basic properties of this relationship are collected in the next lemma:

Lemma 5.2. *Let $A, A', A'', B, B' \in R$-**Noeth**.*

(1) *If $A'' \le A' \le A$, then $A'' \trianglelefteq A$ if and only if $A'' \trianglelefteq A'$ and $A' \trianglelefteq A$.*
(2) *$\psi\colon A \to B$ and $B' \trianglelefteq B \implies \psi^{-1}(B') \trianglelefteq A$*
(3) *$A', A'' \trianglelefteq A \implies A' \cap A'' \trianglelefteq A$*
(4) *$0 \ne A' \trianglelefteq A \implies \operatorname{Kdim} A/A' < \operatorname{Kdim} A$*
(5) *$A' \trianglelefteq A \implies A'$ is essential in A*

Proof. (1) Immediate from the definition.

(2) We have $B' \le B' + \psi(A) \le B$, and so $\operatorname{len} B' = \operatorname{len}(\psi(A) + B') = \operatorname{len} B$. Now consider the exact sequence

$$0 \to \psi^{-1}(B') \xrightarrow{\sigma} A \oplus B' \xrightarrow{\tau} \psi(A) + B' \to 0$$

where $\sigma(a) = (a, \psi(a))$ and $\tau(a, b) = \psi(a) - b$ for all $a \in A$ and $b \in B'$. Using 4.1(1) and 4.4(1), we get

$$\begin{aligned}\operatorname{len} A \oplus \operatorname{len} B' &= \operatorname{len}(A \oplus B') \\ &\le \operatorname{len}(\psi^{-1}(B')) \oplus \operatorname{len}(\psi(A) + B') = \operatorname{len}(\psi^{-1}(B')) \oplus \operatorname{len} B'.\end{aligned}$$

Cancellation from this inequality yields $\operatorname{len} A \le \operatorname{len}(\psi^{-1}(B'))$. Since $\psi^{-1}(B') \le A$, the opposite inequality is, of course, true and we have $\operatorname{len} A = \operatorname{len}(\psi^{-1}(B'))$.

(3) Apply 2 to the inclusion map $\psi\colon A' \to A$.

(4) From 4.1 we get $\operatorname{len}(A/A') + \operatorname{len} A \le \operatorname{len} A$, and then 2.2(1) implies $\operatorname{Kdim} A/A' < \operatorname{Kdim} A$.

(5) If $A' \oplus B \le A$, then $\operatorname{len} A' \oplus \operatorname{len} B = \operatorname{len}(A' \oplus B) \le \operatorname{len} A$. Since $\operatorname{len} A' = \operatorname{len} A$, we can cancel from this inequality to get $\operatorname{len} B = 0$, that is, $B = 0$. □

Essential submodules are not necessarily big — any finite length module which has a proper essential submodule serves as an example.

For the remainder of this section, we will suppose that R is a left Noetherian ring. If $A \in R$-**Noeth**, then A is finitely generated and so, from 4.1(2), $\operatorname{Kdim} A \leq \operatorname{Kdim} R$. For any $a \in A$ we will write $\phi_a : R \to Ra$ for the homomorphism defined by $r \mapsto ra$. Of course, since $R \in R$-**Noeth**, we can apply the results in the previous section to the exact sequence $0 \to \operatorname{ann} a \to R \xrightarrow{\phi_a} Ra \to 0$.

By definition, any nonzero submodule of a critical module is big. This fact has some easy consequences for critical left ideals:

Lemma 5.3. *Let I, J be critical left ideals in a left Noetherian ring R.*

(1) *If $IJ \neq 0$ and $\operatorname{len} I \leq \operatorname{len} J$, then $\operatorname{len} I = \operatorname{len} J$ and there is some $x \in J$ such that ϕ_x is injective on I, $I \cong Ix \trianglelefteq Rx \trianglelefteq J$ and $I \oplus \operatorname{ann} x \trianglelefteq R$.*
(2) *If $I^2 \neq 0$ then there is some $x \in I$ such that ϕ_x is injective on I, $I \cong Ix \trianglelefteq Rx \trianglelefteq I$ and $I \oplus \operatorname{ann} x \trianglelefteq R$.*
(3) *If I is nil, then $I^2 = 0$.*

Proof. (1) Let $x \in J$ be chosen so that $0 \neq Ix \leq J$. Since J is critical we have $\operatorname{len} J = \operatorname{len} Ix = \operatorname{len} Rx$. Since $Ix = \phi_x(I)$ is an image of I, we also have $\operatorname{len} Ix \leq \operatorname{len} I$ and so $\operatorname{len} Ix = \operatorname{len} I = \operatorname{len} Rx = \operatorname{len} J$. From 4.2 we have that ϕ_x is injective on I so $I \cong Ix$, $\operatorname{ann} x \cap I = 0$ and $\operatorname{ann} x \oplus I \leq R$.

From the exact sequence $0 \to \operatorname{ann} x \to R \to Rx \to 0$ we get $\operatorname{len} R \leq (\operatorname{len} Rx) \oplus (\operatorname{len} \operatorname{ann} x) = \operatorname{len}(I \oplus \operatorname{ann} x)$ and so $I \oplus \operatorname{ann} x \trianglelefteq R$.
(2) The special case of 1 when $I = J$.
(3) Suppose, contrary to the claim, that $I^2 \neq 0$. Then, from 2, there is some $x \in I$ such that ϕ_x is an injective map from I to I. But this is impossible since $x^n = 0$ for some $n \in \mathbb{N}$, and hence $\phi_x^n = 0$. □

If I is a critical left ideal in a semiprime left Noetherian ring R, then $I^2 \neq 0$ and so from 5.3(2), there is some $x \in I$ such that $I \cong Ix \trianglelefteq Rx \trianglelefteq I$ and $I \oplus \operatorname{ann} x \trianglelefteq R$. In the next theorem we extend this result to all left ideals of R.

Theorem 5.4. *Let R be a semiprime left Noetherian ring, and I a left ideal such that $\operatorname{len} I = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_n}$ in long normal form. Then there are $x_1, x_2, \ldots, x_n \in I$ such that*

(1) *$Rx_1 \oplus Rx_2 \oplus \ldots \oplus Rx_n \trianglelefteq I$*
(2) *$\operatorname{len} Rx_i = \omega^{\gamma_i}$ for $i = 1, 2, \ldots, n$.*
(3) *$x_i x_j = 0$ whenever $i < j$ with $i, j = 1, 2, \ldots, n$.*

Setting $x = x_1 + x_2 + \ldots + x_n$ we also have

4. *ϕ_x is injective on I*

5. $I \cong Ix \trianglelefteq Rx \trianglelefteq I$
6. $I \oplus \operatorname{ann} x \trianglelefteq R$

Proof. Proof by induction on $n = \operatorname{Krank} I$.

From 4.6(2), the left ideal I contains a critical left ideal I_n of length ω^{γ_n}. Since R is semiprime, $I_n^2 \neq 0$, and from 5.3(2) there is some $x_n \in I_n$ such that $\operatorname{len} I_n x_n = \operatorname{len} I x_n = \operatorname{len} R x_n = \omega^{\gamma_n}$ and $\operatorname{ann} x_n \cap R x_n = 0$.

Let $I' = \operatorname{ann} x_n \cap I$. Then $I' \oplus R x_n \leq I$, so that $\operatorname{len} I' \oplus \omega^{\gamma_n} \leq \operatorname{len} I$. From the obvious short exact sequence $0 \to I' \to I \to I x_n \to 0$ we get $\operatorname{len} I \leq \operatorname{len} I' \oplus \operatorname{len} I x_n = \operatorname{len} I' \oplus \omega^{\gamma_n}$. Thus, in fact, $\operatorname{len} I = \operatorname{len} I' \oplus \omega^{\gamma_n}$. Canceling ω^{γ_n} from this equation we get $\operatorname{len} I' = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_{n-1}}$, and so, $\operatorname{Krank} I' = n - 1 < \operatorname{Krank} I = n$.

By induction there are $x_1, x_2, \ldots, x_{n-1} \in I'$ satisfying the above conditions with respect to I'. We claim that $x_1, x_2, \ldots, x_n$ satisfy these conditions with respect to I:

By induction we have $I' \cap \operatorname{ann}(x_1 + x_2 + \ldots + x_{n-1}) = 0$. We also have $R(x_1 + x_2 + \ldots + x_{n-1}) \cap R x_n \subseteq I' \cap R x_n = 0$, from which it follows that $\operatorname{ann} x = \operatorname{ann}(x_1 + x_2 + \ldots + x_{n-1}) \cap \operatorname{ann} x_n$. A simple calculation then yields $I \cap \operatorname{ann} x = 0$. Claims 4, 5 and 6 follow from this as in the proof of 5.3(1).

The remaining claims are easy to check. ☐

Corollary 5.5. *Let R be a semiprime left Noetherian ring, $I \leq R$ a left ideal and $r \in R$.*

1. $\operatorname{ann} r \trianglelefteq R \iff r = 0$
2. $\operatorname{ann} r = 0 \iff Rr \trianglelefteq R \iff r$ *is regular.*
3. *I is essential in R* $\iff I \trianglelefteq R \iff$ *I contains a regular element.*
4. *If I is nil, then $I = 0$.*
5. $\operatorname{udim} I = \operatorname{Krank} I$.

Proof.
1. Applying 5.4(6) to the left ideal Rr, we see that there is some $s \in R$ such that $Rr \cap \operatorname{ann} sr = 0$. But $\operatorname{ann} sr = \phi_s^{-1}(\operatorname{ann} r)$, so from 5.2(2), we have $\operatorname{ann} sr \trianglelefteq R$, and, in particular, $\operatorname{ann} sr$ is essential in R. Thus $Rr = 0$ and $r = 0$.
2. Suppose $Rr \trianglelefteq R$. Then by 4.2, the homomorphism ϕ_r is injective and hence $\operatorname{ann} r = 0$. Further, if $rs = 0$ for some $s \in R$, then $Rr \leq \operatorname{ann} s$, and so $\operatorname{ann} s \trianglelefteq R$ and then, by 1, $s = 0$. Thus r is regular. The remaining claims are easy.
3. If I is essential, then, from 5.4(6), I contains an element x such that $\operatorname{ann} x = 0$. From 2, x is regular. The remaining claims are easy.
4. From 5.4(4), there is some $x \in I$ such that ϕ_x is injective on I. Since $x^n = 0$ for some $n \in \mathbb{N}$, we have $\phi_x^n = 0$ and hence $I = 0$.

   For an alternative proof which avoids 5.4, notice first that if I is critical, then from 5.3(3), $I^2 = 0$, and then, because R is semiprime,

$I = 0$. Thus R has no critical nil left ideals. The claim then follows from the fact that any nonzero nil left ideal must contain a critical nil left ideal.

(5) From 5.4(1), I contains a direct sum of $\operatorname{Krank} I$ nonzero submodules, and so $\operatorname{Krank} I \leq \operatorname{udim} I$. The opposite inequality is 4.5(3). □

Notice also that, from 5, I is uniform if and only if it is critical.

We now specialize to left Noetherian prime rings.

Theorem 5.6. *If R is a left Noetherian prime ring, then $\operatorname{len}_R R = \omega^\gamma n$ where $\gamma = \operatorname{Kdim}_R R$ and $n = \operatorname{udim}_R R$. Further, for $A \in R$-**Noeth** and $m \in \mathbb{N}$ we have*

(1) *$\omega^\gamma m \leq \operatorname{len} A$ if and only if A has a submodule isomorphic to a direct sum of m critical left ideals.*

(2) *$(\operatorname{len}_R R)m \leq \operatorname{len} A$ if and only if A has a submodule isomorphic to R^m.*

Proof. First we notice that for any two critical left ideals $I, J \leq R$ we have $IJ \neq 0$ and $JI \neq 0$ and so from 5.3(1), $\operatorname{len} I = \operatorname{len} J$, I has a submodule isomorphic to J, and vice versa. In this situation I and J are said to be **subisomorphic** [17, 3.3.4].

If $\operatorname{len}_R R = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_n}$ in long normal form, then from 5.4(1), there are critical left ideals of length $\omega^{\gamma_1}, \omega^{\gamma_2}, \ldots, \omega^{\gamma_n}$. From above, we must have have $\gamma_1 = \gamma_2 = \ldots = \gamma_n$ and so we can write $\operatorname{len}_R R = \omega^\gamma n$ as required. This means, in particular, that any critical left ideal of R has length ω^γ.

(1) Proof by induction on m, the case $m = 0$ being trivial.

Suppose $0 < m$ and $\omega^\gamma m \leq \operatorname{len} A$. Then by 4.6(1) there is some submodule $A' \leq A$ such that $\operatorname{len} A/A' = \omega^\gamma$. Using 4.1(1), we have $\omega^\gamma m \leq \operatorname{len} A \leq \operatorname{len}(A/A') \oplus \operatorname{len} A' = \omega^\gamma \oplus \operatorname{len} A'$, so by cancellation $\omega^\gamma(m-1) \leq \operatorname{len} A'$. By induction, A' contains a submodule isomorphic to a direct sum of $m-1$ critical left ideals.

Let $a \in A \setminus A'$. Then $\operatorname{len}(Ra + A')/A' = \omega^\gamma$. From the exact sequence

$$0 \to \operatorname{ann}(a + A') \to R \to (Ra + A')/A' \to 0$$

and 4.1(1) we get $\operatorname{len}((Ra + A')/A) + \operatorname{len}(\operatorname{ann}(a + A')) \leq \operatorname{len} R$, that is, $\omega^\gamma + \operatorname{len}(\operatorname{ann}(a + A')) \leq \omega^\gamma n$. Cancellation from this inequality gives $\operatorname{len}(\operatorname{ann}(a + A')) \leq \omega^\gamma(n-1) < \operatorname{len} R$.

From 5.5(3), $\operatorname{ann}(a + A')$ is not essential in R, and there is a critical left ideal I of R such that $I \cap \operatorname{ann}(a + A') = 0$. The map

$\phi_{a+A'}\colon I \to (Ia + A')/A'$ is then an isomorphism. In particular $\phi_{a+A'}$ is injective on I, so for any $u \in I$, $ua \in A'$ implies $u = 0$. Thus $Ia \cap A' = 0$, and $Ia \cong Ia + A'/A' \cong I$ is critical. Since $Ia \cap A' = 0$, A contains a direct sum of m critical modules.

(2) In view of 1, to prove 2, it suffices to show that any (external) direct sum of n critical left ideals, contains a submodule isomorphic to R. Since critical left ideals are pairwise subisomorphic, it suffices to show this for any particular direct sum of n critical left ideals. Now, from 5.4, there are critical left ideals $I_1, I_2, \dots, I_n$ such that $I_1 \oplus I_2 \oplus \dots \oplus I_n \trianglelefteq R$, and $x \in I_1 \oplus I_2 \oplus \dots \oplus I_n$ such that $Rx \cong R$. Thus this particular direct sum contains a submodule isomorphic to R as required. □

We will next show that, for a Noetherian module A over a left Noetherian prime ring, $\operatorname{len} A$ encodes the reduced rank $\rho(A)$ [16, 3.5.4] [19, 6.3], and also whether or not A is torsion or torsion free.

By definition, an element $a \in A$ is **torsion** if $\operatorname{ann} a$ is essential in R. From 5.5(3), a is torsion if and only if $\operatorname{ann} a \trianglelefteq R$. Applying 4.1 and 2.2 to the exact sequence $0 \to \operatorname{ann} a \to R \to Ra \to 0$ we get a is torsion if and only if $\operatorname{Kdim} Ra < \operatorname{Kdim} R$, if and only if $\operatorname{len} Ra < \omega^\gamma$, if and only if $\operatorname{len}_\gamma Ra = 0$, where $\gamma = \operatorname{Kdim} R$. The module A is **torsion** if all its elements are torsion, and is **torsion free** if 0 is the only torsion element.

Corollary 5.7. *Let R be a left Noetherian prime ring, $\gamma = \operatorname{Kdim}{}_R R$, and $A \in R$-***Noeth***.*

(1) *A is torsion if and only if $\operatorname{Kdim} A < \gamma$.*
(2) *A is torsion free if and only if $\operatorname{len} A = \omega^\gamma k$ for some $k \in \mathbb{Z}^+$.*
(3) *$\rho(A) = \operatorname{len}_\gamma A$.*

Proof. (1) If $\operatorname{Kdim} A < \gamma$, then for any $a \in A$, we have $\operatorname{Kdim} Ra < \gamma$ and hence, from the preceding discussion, a is torsion.

If A is torsion with generators $a_1, a_2, \dots, a_n$, then $\operatorname{Kdim} Ra_i < \gamma$ for $i = 1, 2, \dots, n$. From 4.3(4), $\operatorname{Kdim} A = \max_i\{\operatorname{Kdim} Ra_i\} < \gamma$.

(2) Follows from 1, since, $\operatorname{Kdim} A \le \operatorname{Kdim} R = \gamma$ and then, by 4.6, A has no nonzero submodule with Krull dimension less than γ if and only if $\operatorname{len} A = \omega^\gamma k$ for some $k \in \mathbb{Z}^+$.

(3) Set $k = \operatorname{len}_\gamma A$. Since $\operatorname{Kdim} A \le \gamma$ we have $\omega^\gamma k \le \operatorname{len} A < \omega^\gamma(k + 1)$. From 5.6(1), A contains a submodule A' isomorphic to a direct sum of k critical (and hence uniform) left ideals of R. Since uniform left ideals have reduced rank 1 [19, 6.11(f)] we have $\rho(A') = k$.

From 4.1(1), we have $\operatorname{len} A/A' + \operatorname{len} A' \le \operatorname{len} A$ and consequently $\operatorname{len} A/A' + \omega^\gamma k \le \operatorname{len} A < \omega^\gamma(k + 1)$. From 2.2(3), $\operatorname{len} A/A' < \omega^\gamma$

and hence $\operatorname{Kdim} A/A' < \gamma$. From 1, A/A' is a torsion module and so $\rho(A/A') = 0$. Since reduced rank respects short exact sequences we have $\rho(A) = \rho(A') + \rho(A/A') = k$. □

There are two obvious limits on the length of a cyclic module Ra contained in $A \in R$-**Noeth**: We must have $\operatorname{len} Ra \leq \operatorname{len} A$ and $\operatorname{len} Ra \leq \operatorname{len} R$, that is, $\operatorname{len} Ra \leq \min\{\operatorname{len} A, \operatorname{len} R\}$.

From Theorem 5.6(2) we have that if R is a left Noetherian prime ring, and $A \in R$-**Noeth** is such that $\operatorname{len} R \leq \operatorname{len} A$, then A contains a cyclic module Ra with $\operatorname{len} Ra = \operatorname{len} R$. So in this case the maximum possible length is attained.

When $\operatorname{len} A < \operatorname{len} R$, there is no guarantee that A has a cyclic submodule of length $\operatorname{len} A$ — any noncyclic finite length module serves as an example. On the other hand, we will show that, when R is a simple ring, any module $A \in R$-**Noeth** contains a cyclic submodule of maximum possible length.

In the proof of this result we use, not just the fact that any nonzero Noetherian module contains a critical submodule, but also the special arithmetic properties of the lengths of the critical submodule and the factor module. Specifically, suppose A is a nonzero Noetherian module with $\operatorname{len} A = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_n}$ in long normal form. Set $\beta = \omega^{\gamma_n}$ and $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_{n-1}}$. Then from 3.4, we have $\alpha + \beta = \alpha \oplus \beta$, and from 4.6(2), there is a critical submodule $A' \leq A$ such that $\operatorname{len} A' = \beta$ and $\operatorname{len} A/A' = \alpha$. Note that $\operatorname{Krank} A/A' < \operatorname{Krank} A$. Moreover, if $B' \leq B$ are any modules such that $\operatorname{len} B' = \beta$ and $\operatorname{len} B/B' = \alpha$, then, by 4.1(1), $\operatorname{len} B = \alpha + \beta = \alpha \oplus \beta = \operatorname{len} A$.

Theorem 5.8. *If R is a simple left Noetherian ring and $A \in R$-**Noeth**, then there is some $a \in A$ such that* $\operatorname{len} Ra = \min\{\operatorname{len} A, \operatorname{len} R\}$.

Proof. We prove the claim by induction on $\operatorname{Krank} A$.

Let $A' \leq A$ be a critical submodule as described above with $\operatorname{len} A' = \beta$, $\operatorname{len} A/A' = \alpha$, and $\operatorname{Krank} A/A' < \operatorname{Krank} A$. By induction, there is some $a \in A$ such that $\operatorname{len}(Ra + A')/A' = \operatorname{len} A/A'$.

If $\operatorname{ann} a = 0$, then $Ra \cong R$, so $\operatorname{len} R = \operatorname{len} Ra \leq \operatorname{len} A$ and we have proved the claim.

This leaves us the case that $\operatorname{ann} a \neq 0$. Because R is simple, we have $\operatorname{ann} A' = 0$, and there is some $r \in R$ and $a' \in A'$ such that $ra = 0$ but $ra' \neq 0$.

Set $B = R(a + a')$ and $B' = R(a + a') \cap A' \leq B$. Then

$$\begin{aligned}\operatorname{len} B/B' &= \operatorname{len}(R(a + a')/(R(a + a') \cap A'))\\ &= \operatorname{len}(R(a + a') + A'/A')\\ &= \operatorname{len}(Ra + A'/A') = \alpha\end{aligned}$$

and $0 \neq ra' = r(a+a') \in R(a+a') \cap A' = B'$ so $\operatorname{len} B' = \operatorname{len} A' = \beta$. Hence from the above discussion $\operatorname{len} R(a + a') = \operatorname{len} B = \operatorname{len} A$. □

Of course, if $\operatorname{len} A \leq \operatorname{len} R$ in this theorem, then $Ra \trianglelefteq A$, so a simple special case of this theorem is the following: If A is a finite length module over a simple left Noetherian ring R such that $\operatorname{len} A \leq \operatorname{len} R$, then A is a cyclic module. That is, A has a single generator.

This observation generalizes immediately to modules A which have finite Krull dimension, because if $Ra \trianglelefteq A$, then, from 5.2(4), A/Ra has strictly smaller Krull dimension than A. A simple induction argument then yields

Corollary 5.9. *If R is a simple left Noetherian ring and $A \in R$-**Noeth** is such that $\operatorname{len} A \leq \operatorname{len}_R R$ and $\operatorname{Kdim} A < \omega$, then A can be generated by $\operatorname{Kdim} A + 1$ elements.*

This corollary is a version of two related results of Stafford [20]: 1) Any left ideal of a simple left Noetherian ring with finite Krull dimension can be generated by $\operatorname{Kdim} R + 1$ elements, and 2) Any Noetherian torsion module A with finite Krull dimension over a simple left Noetherian ring can be generated by $\operatorname{Kdim} A + 1$ elements.

Since a simple ring is also a prime ring, we can use 5.6(2) to calculate bounds on the number of generators for larger modules. For example, if R is a simple left Noetherian ring with $\operatorname{len} R = \omega^3 4$ and $A \in R$-**Noeth** with $\operatorname{len} A = \omega^3 9 + \omega^2 + 1$. Then $(\operatorname{len} R)2 \leq \operatorname{len} A$ so from 5.6 we know that A has a submodule A' isomorphic to R^2. In particular, A' has 2 generators and length $\omega^3 8$.

We have $\operatorname{len} A/A' + \operatorname{len} A' \leq \operatorname{len} A < \omega^3 10$, that is, $\operatorname{len} A/A' + \omega^3 8 < \omega^3 10$. From 2.2(3) we get $\operatorname{len} A/A' < \omega^3 2$ and then, from 5.9, we know that A/A' has 4 generators. Since A' has two generators, A has 6 generators.

We leave the reader the task of checking the details in the following generalization of the above argument which has been expressed in terms of reduced rank using 5.7:

Theorem 5.10. *Let R be a simple left Noetherian ring with finite Krull dimension. For $A \in R$-**Noeth**, define*

$$b(A) = \begin{cases} \operatorname{Kdim} A + 1 & \text{if } \rho(A) = 0 \\ \operatorname{Kdim} R + \lceil \rho(A)/\rho(R) \rceil & \text{otherwise} \end{cases}$$

where $\lceil \rho(A)/\rho(R) \rceil$ *is the smallest natural number greater than or equal to* $\rho(A)/\rho(R)$*. Then* A *can be generated by* $b(A)$ *elements.*

The reader can check that this theorem and Corollary 5.9 are very special cases of the Stafford-Coutinho Theorem [16, 3.5.72] on the stable number of generators of Noetherian modules.

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