Lesson Plan: Using the Formal Definition of Limits

Sam Chyau
Department of Mathematics
California State University, Los Angeles

Lesson: Constructing a Formal Argument for a Limit.

Timeframe: Approximately 60-80 minutes for pre-class activities; 75 minutes in class.

Target Audience: Calculus I students

Materials: Course textbook, four online videos, pre-class worksheet, in-class exercises, and homework set.

Basic Objectives: Upon completing the pre-class activities, students will be able to:

- Recite the formal definition of a limit.
- Explain the relationship between the informal description of a limit and the formal definition.
- Given a graph of a simple function continuous at $x = c$, informally describe why the necessary value of δ decreases as ϵ decreases in accordance with the formal definition of limit.

Advanced Objectives: Upon completing this lesson, students will be able to:

- Given a real value c and a real-valued function f, determine a positive δ that satisfies the formal definition of a limit for different values of positive ϵ.
- Compose a proof of a limit using the formal definition for certain simple functions.
- Given a function f and a real value c such that the limit of f as x approaches c is *not* L, find an ϵ for which no positive δ satisfies the formal definition of limit.

Background: Students participating in this lesson have already been exposed to an informal characterization of limits in a previous section of the text, and they have done many exercises in calculating limits by observing graphs and by using formulas. They have also done exercises in which they identify when a limit as x approaches a value c either does not exist or does not match the output of the function at $x = c$.

Pre-Class Activity - Introduction and Preliminary Exposure: Students will complete the following activity prior to the class meeting. *Approximately 60-80 minutes.*

1. Watch the first three videos on https://www.khanacademy.org/math/calculus-home/limits-and-continuity-calc/formal-definition-of-limits-calc/v/proving-a-limit-using-epsilon-delta-definition (3:35, 6:06, and 6:58, respectively).
   - The first video is sufficient review/revisitation of the informal definition of limits that students have previously seen.
   - The second and third videos develop the formal definition of limits. Emphasize to students that the primary concepts of the lesson lie in Videos 2 and 3.
2. Answer the following:
   (a) Why was our earlier description of limits not so mathematically precise?
   (b) What mathematical inequality indicates "$f(x)$ is within ϵ of L" in the definition of the limit?
   (c) What mathematical inequality indicates "x is within δ of c" in the definition of the limit?
   (d) Formally, what is the definition of the limit?
   (e) Why is it not necessary for f to be *defined* at c in order for the limit as x approaches c to exist?
3. Watch the fourth video on the site provided (8:29).
4. Answer the following:
   (a) Explain why δ is a function of ϵ in the precise definition of the limit.
   (b) Using the function provided in the video, what value of δ would be small enough to satisfy the definition of the limit when $\epsilon = 2$? When $\epsilon = 0.4$? When $\epsilon = 0.001$?
   (c) Let $g(x) = 3x$ when $x \neq 4$ and 23 otherwise. Graph the function, and argue using the formal definition that the limit of $g(x)$ as x approaches 4 is 12.
   (d) Let $h(x) = 1$ when $x \geq 0$ and -1 otherwise. Graph the function. What part of the precise definition of limits fails for $h(x)$ as x approaches 0?

In-Class Activities - Discussion, Higher Examples, and Practice: *75 minutes total.*

1. Form groups (each with 3-4 students) and have students share results from the pre-class activity with group members.
2. Ask students to describe *how* they found their answers on question 4(b). What is happening to the values of δ as ϵ decreases?

3. Bring two students to the front of the class to share their proofs on question 4(c). Discuss with the class any errors in the argument with particular focus on structure and organization.

4. Have one student share their response on 4(d) with the class and explain their reasoning. The rest of the students in the class will point out any faults in the presenting student's reasoning, and the instructor will show how to remedy such faults.

5. Present to the class slightly more difficult proofs of limits:

$$\lim_{x \to 6} \left(\frac{12x^2 - 40x - 7}{2x - 7} \right) = 37$$

   and

$$\lim_{x \to -3} \left((x+3)^6 - 5 \right) = -5$$

   Iterate in particular the process of *finding* the appropriate value of δ for a given value of ϵ. In these examples, the instructor does the problem on the board with student guidance.

6. Have students practice independently proving the following using the formal definition of limits:

$$\lim_{x \to 9} \left((2x-18)^4 + 11 \right) = 11$$

   and

$$\lim_{x \to -7} \left(\frac{6x^2 - 23x - 18}{3x + 2} \right) = -23$$

   As students work on problems, be sure to do the following:

   - Monitor students' scratch work for preliminary assessment of student understanding of the definition and the goal of finding δ.
   - Identify and remark on student misconceptions whenever seen.
   - Provide small hints or reminders where warranted.

7. Have students share their responses with group members for critique. Students should focus on:

   - Structure and organization of arguments.
   - Using precise terminology in arguments.
   - Finding a valid δ to use.
   - Showing that the proposed δ satisfies the definition.

8. Provide and explain the solutions to the two practice problems. Explain each step of calculating δ with particular detail (skip no steps in the arithmetic/algebra).

Closure - Reflection and Independent Practice

1. Provide opportunity for last-minute student questions.
2. Provide a 3-minute reflection of the day's activities and the abstractions of the formal definition.
3. Ask students to think about why proving the following would be much harder than the examples from class:
$$\lim_{x \to 5} \left(x^3 + 1\right) = 126$$
4. Assign practice problems from the course text for independent home practice. Problems will be collected in the following class meeting and assessed.

Analysis of Lesson: The primary strength of this lesson is that it exercises independent abstract student thinking, attention to mathematical definition, trial and error, and most importantly, formal argument. For the instructor, the challenge lies in the requirement of formatively assessing all (or at least most) students' understanding during in-class practice, since student misconceptions in the definition are likely to aplenty within the class.