California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Linear Analysis (old version)   Spring 2021

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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Fall 2020 #1. We define a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = Cx + b,$$

where $C = (c_{ij})$ is a real $n \times n$ matrix and $b \in \mathbb{R}^n$ is given.

(a) If we equip $\mathbb{R}^n$ with the metric $d(x, y) = \max_{1\le i\le n} |x_i - y_i|$, under what general condition on the matrix C will T be a proper contraction? Justify your answer.

(b) Repeat Part (a) with the metric $d(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{1/2}$.

(c) With either of the metric from (a) or (b), describe an iteration process for solving the linear system of equations

$$x = Cx + b$$

by specifying the transformation to be iterated and explaining how this leads to a solution.

Fall 2020 #2. Let $p_0(x) = 1$ and $p_1(x) = x$, and let $\mathcal{P}_1$ be the subspace of the space $C([0,2])$ of all continuous functions on $[0,2]$ spanned by p_0 and p_1.

(a) Find a basis for $\mathcal{P}_1$ which is orthonormal with respect to the inner product $\langle f, g\rangle = \int_0^2 f(t)\overline{g(t)}\,dt$.

(b) Use the results of Part (a) to find the function $f(x) = ax + b$ in $\mathcal{P}_1$ which makes the quantity

$$J(f) = \int_0^2 |x^2 - f(x)|^2\,dx$$

as small as possible.

Fall 2020 #3. If d is a metric on a vector space $X \neq \{0\}$ which is obtained from a norm $\|\cdot\|$, and $\tilde{d}$ is defined by

$$\tilde{d}(x,x) = 0, \qquad \tilde{d}(x,y) = d(x,y) + 1 \quad (x \neq y).$$

(a) Show that $\tilde{d}$ is a metric on X.

(b) Show that $\tilde{d}$ cannot be obtained from a norm.

Fall 2020 #4. Let $C^2(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the second derivative, f'', exists and is continuous. For f in $C^2(\mathbb{R})$, let $(Lf)(t) = f''(t) - 2f(t)$.

(a) Show that $C^2(\mathbb{R})$ is a vector subspace of $C(\mathbb{R})$, the space of all continuous real valued functions on $\mathbb{R}$.

(b) Show that L is a linear transformation from $C^2(\mathbb{R})$ into $C(\mathbb{R})$.

(c) Let $\mathcal{W} = \{f \in C^2(\mathbb{R}) \mid f''(t) - 2f(t) = 0 \text{ for all } t \in \mathbb{R}\}$. Show that $\mathcal{W}$ is a vector subspace of $C^2(\mathbb{R})$.

(d) Suppose we change the criterion in Part (c) to $f''(t) - 2f(t) = \sin(t)$. Now is $\mathcal{W}$ a vector subspace? Why or why not?

Fall 2020 #5. Let $f(t) = t^2$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Find the Fourier series for $f(t)$ in the trigonometric form.

(b) Use the result of Part (a) to show that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Fall 2020 #6. Let $E = \{f : [-\pi, \pi] \to \mathbb{R} \mid \int_{-\pi}^{\pi} f^2(x)\,dx < \infty\}$. The inner product on E is defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

We define an operator $\mathcal{A}$ on E by

$$(\mathcal{A}f)(x) = \int_{-\pi}^{\pi} k(x, y)f(y)\,dy$$

with $k(x, y) = \sin(x + y)$.

(a) Prove that $\mathcal{A}$ is continuous from E to $C([-\pi, \pi])$, the space of continuous functions, equipped with the sup norm.

(b) Is $\mathcal{A}$ self-adjoint from E to $C([-\pi, \pi])$? Justify your answer.

(c) Is $\mathcal{A}$ compact from E to $C([-\pi, \pi])$? Justify your answer.

Fall 2020 #7. Let T_r and T_l be two operators such that for any $x \in \ell^2(\mathbb{C})$,

$$T_r(x) = (0, x_1, x_2, \ldots, x_n, \ldots), \quad T_l(x) = (x_2, x_3, \ldots, x_n, \ldots),$$

where $\ell^2(\mathbb{C}) = \{x = (x_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$.

(a) Show that T_r and T_l are two linear bounded operators on $\ell^2(\mathbb{C})$.

(b) Find the operator norms $\|T_r\|$ and $\|T_l\|$.

(c) Find $T_r \circ T_l$ and $T_l \circ T_r$.

(d) Find the adjoint operator of T_l.