California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Fall 2001

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Do **five** of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}$ denotes the set of real numbers.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

$\bar{z}$ denotes the complex conjugate of the complex number z.

$|z|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2 \, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\, dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Fall 2001 # 1. Let (x, y) and (a, b) represent points in $\mathbb{R}^2$.

a. For each of the following decide whether the formula given for $\| (x, y) \|$ defines a norm on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.

(i) $\| (x, y) \| = 2\,|x| + 3\,|y|$
(ii) $\| (x, y) \| = x^2 + y^2$

b. For each of the following decide whether the formula given for $\langle\, (a, b), (x, y) \,\rangle$ defines an inner product on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.

(i) $\langle\, (a, b), (x, y) \,\rangle = 2ax + 3by$
(ii) $\langle\, (a, b), (x, y) \,\rangle = 2ax - 3by$

Fall 2001 # 2. Let $\mathcal{P}^2$ be the space of all polynomials of degree no more than 2 with the inner product $\langle\, f, g \,\rangle = \displaystyle\int_{-1}^{1} f(t)\overline{g(t)}\, dt$ and the associated norm.

For f in $\mathcal{P}^2$, let $\phi(f) = f'(0)$.

a. Show that ϕ is a linear functional on $\mathcal{P}^2$.
b. Find a polynomial q in $\mathcal{P}^2$ such that $\phi(f) = \langle\, f, q \,\rangle$ for all f in $\mathcal{P}^2$.
c. Show that ϕ is continuous with respect to the specified norm on $\mathcal{P}^2$.
d. Find the operator norm of ϕ as a linear functional on $\mathcal{P}^2$ (with the specified norm).

Fall 2001 # 3. Consider the boundary value problem

$$(*) \qquad \frac{d}{dx}\left[\frac{1}{x^2}\frac{dy}{dx}\right] = f(x) \quad \text{for } 1 \le x \le 2 \qquad \textit{with} \quad y'(1) = 0 \text{ and } y(2) = 0.$$

a. Find G(x,t) such that the solutions to (*) for known function f are given by

$$y(x) = \int_1^2 G(x, t) f(t)\, dt$$

b. Use the function found in part **a** to solve the boundary value problem (*) with $f(x) = 3$ for all x.

Fall 2001 # 4. Let $\mathcal{H} = \{f \in C([0, \pi]) : f(0) = 0\}$ with the inner product $\langle\, f, g \,\rangle = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(t)\overline{g(t)}\, dt$.

For $n = 1, 2, 3, \ldots$ let $s_n(x) = \sin nx$.

Let $\mathcal{S} = \{s_1, s_2, s_3, \ldots\}$.

a. Show $\mathcal{S}$ is an orthonormal family in $\mathcal{H}$.
b. Show $\mathcal{S}^{\perp} = \{0\}$

Suggestion: For f in $\mathcal{H}$, define f_o on $[-\pi, \pi]$ by putting $f_o(x) = f(x)$ for $x \ge 0$ and $f_o(x) = -f(-x)$ for $x < 0$. Then use your knowledge of the trigonometric family of functions $\mathcal{T} = \{1/\sqrt{2}, \cos x, \sin x \cos 2x, \sin 2x, \ldots\}$ on $[-\pi, \pi]$

Fall 2001 # 5. For$f$ in $L^2[(0,1])$, define Kf by

$$(Kf)(x) = \int_0^1 (1 - 3xt) f(t)\, dt.$$

a. Find any nonzero eigenvalues and the associated eigenfunctions for the integral operator K.

b. Find a function $R(x,t;\lambda)$ such that solutions f to the integral equation

$$f(x) = g(x) + \lambda \int_0^1 (1 - 3xt) f(t)\, dt$$

for known function g are given by

$$f(x) = g(x) + \lambda \int_0^1 R(x,t;\lambda) g(t)\, dt \tag{*}$$

c. Solve the equation (*) when $\lambda = 1$ and $g(x) = 1$ for all x.

Fall 2001 # 6. Let $\mathcal{P}^1$ be the space of all polynomials of degree no more than 1 with the inner product $\langle f, g \rangle = \displaystyle\int_0^2 f(t)\overline{g(t)}\, dt$

a. Find a basis for $\mathcal{P}^1$ which is orthonormal with respect to that inner product

b Find constants a and b which make the quantity $J = \displaystyle\int_0^2 \left|t^2 - a - bt\right|^2 dt$ as small as possible.

Fall 2001 # 7. For each continuous function f on the interval $[01, 1]$ define a function Tf by

$$(Tf)(x) = x - \lambda \int_0^x (x - t) f(t)\, dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm. Justify your answer.

b. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the L^2 norm. Justify your answer.

c. Describe the iterative process for solving the integral equation

$$f(x) = x - \lambda \int_0^x (x - t) f(t)\, dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

End of Exam