California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Fall 2002

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Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a, b])$ denotes the space of all continuous functions on the interval $[a, b]$. If there is need to specify the possible values, $\mathcal{C}([a, b], \mathbb{R})$ will denote the space of all continuous real valued functions on $[a, b]$ and $\mathcal{C}([a, b], \mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a, b])$ denotes the space of all functions on the inteval $[a, b]$ such that $\int_a^b |f(x)|^2 \, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \qquad \cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b) \qquad 2 \cos a \cos b = \cos(a - b) + \cos(a + b)$$

$$2 \sin a \cos b = \sin(a + b) + \sin(a - b) \qquad 2 \cos a \sin b = \sin(a + b) - \sin(a - b)$$

$$\int \sin^2(ax) \, dx = \frac{x}{2} - \frac{1}{4a} \sin(2ax) \qquad \int \cos^2(ax) \, dx = \frac{x}{2} + \frac{1}{4a} \sin(2ax)$$

$$\int x \sin(ax) \, dx = \frac{1}{a^2} \sin(ax) - \frac{x}{a} \cos(ax) \qquad \int x \cos(ax) \, dx = \frac{1}{a^2} \cos(ax) + \frac{x}{a} \sin(ax)$$

Fall 2002 # 1. Let f be defined on $[-\pi, \pi]$ by $f(x) = \begin{cases} x+\pi, & \text{for } -\pi \le x \le 0 \\ \pi - x, & \text{for } 0 < x \le \pi \end{cases}$

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a. Compute the Fourier series for f on $[-\pi, \pi]$.
(Trigonometric or exponential, your choice)

b. Show that $\quad 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots = \frac{\pi^4}{96}$

Fall 2002 # 2. **a.** Suppose $\langle \cdot, \cdot \rangle$ is an inner product on a vector space $\mathcal{V}$ and $\| \cdot \|$ is the associated norm. Show that if f and g are vectors in $\mathcal{V}$, then $\| f+g \|^2 + \| f-g \|^2 = 2\| f \|^2 + 2\| g \|^2$.

b. Show that there is no possible inner product on the space $C([-\pi, \pi])$ of continuous real valued functions on the interval $[-\pi, \pi]$ for which the uniform norm, $\| f \|_\infty = \sup\{|f(t)| : t \in [-\pi, \pi]\}$, is the associated norm.
(Hint: What happens if one function is 0 when $x \le 0$ and the other when $x \ge 0$?)

Fall 2002 # 3. Let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous real valued functions on the interval $[a, b]$.

a. State definitions for each of the following:

(i) $f_n \to f$ pointwise on $[a, b]$
(i) $f_n \to f$ uniformly on $[a, b]$
(i) $f_n \to f$ with respect to the L^2-norm (that is, in L^2-mean) on $[a, b]$

b. Show that if $f_n \to f$ in L^2-norm, then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

c. Give an example in which $f_n \to f$ pointwise on $[a, b]$ but $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \ne \int_a^b f(x)\,dx$.

d. Show that if $f_n \to f$ uniformly on $[a, b]$, then $f_n \to f$ in L^2-norm on $[a, b]$.

Fall 2002 # 4. **a.** Show that the operator $L = -\frac{d^2}{dx^2}$ acting on the space

$$\mathcal{W} = \{f : [0,1] \to \mathbb{R} : f'' \text{ is continuous }, f(0) = 0, \text{ and } f(1) - f'(1) = 0\}$$

is a symmetric operator with respect to the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

b. Show that if f and g are in $\mathcal{W}$ with $Lf = \lambda f$, $Lg = \mu g$ and $\mu \ne \lambda$, then f and g are orthogonal with respect to that inner product.

c. Show that there are infinitely many positive values of the number λ for which the problem

$$Lf = \lambda f \qquad \text{with} \qquad f(0) = 0 \quad \text{and } f(1) - f'(1) = 0$$

has nonzero solutions f. (You need not find the λ's, but, if you don't, then say something about where they are on the positive real axis.)
(Hint: sketch the graphs of $y = x$ and $y = \tan x$ on the same set of axes.)

Fall 2002 # 5. Suppose $k(x,t)$ is a continuous real valued functin on the square $[a,b]\times[a,b]$ such that $k(x,t)=k(t,x)$ for all x and t in $[a,b]$. For each continuous f on $[a,b]$, let Kf be defined by

$$(Kf)(x)=\int_a^b k(x,t)f(t)\,dt$$

Suppose $\{\phi_j\}_{j=1}^{\infty}$ is a complete orthonormal family of functions on $[a,b]$ with respect to the innerproduct $\langle f,g\rangle=\int_a^b f(t)\overline{g(t)}\,dt$ with $K\phi_j=\mu_j\phi_j$. For continuous g on $[a,b]$ and a nonzero number λ, consider the equation

$$\text{(A)}\qquad f(x)=g(x)+\lambda\int_a^b k(x,t)f(t)\,dt$$

a. Show how to write the solution to equation (A) in terms of g, λ, $\{\phi_j\}_{j=1}^{\infty}$, and $\{\mu_j\}_{j=1}^{\infty}$ if $1/\lambda$ is not one of the μ_j.

b. What happens if $1/\lambda$ is one of the μ_j?

Fall 2002 # 6. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{pmatrix}2\\0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$.

a. Use the Gram-Schmidt process to find an orthonormal basis for $\mathcal{W}$.

b. Find the vector in $\mathcal{W}$ closest to $v=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$. Call it w.

c. What does the Bessel inequality state for the vectors in **(b)** (with numerical values)?

Fall 2002 # 7. For each continuous function f on the interval $[0,1]$, let

$$(Kf)(x)=\int_0^1 f(t)\sin\pi x\sin\pi t\,dt.$$

a. Find a function $R(x,t;\lambda)$ such that the solution to the equation $f=g+\lambda Kf$ is given by

$$f(x)=g(x)+\lambda\int_0^1 R(x,t;\lambda)g(t)\,dt.$$

b. Find a function f such that

$$f(x)=1+\int_0^1 f(t)\sin\pi x\sin\pi t\,dt$$

Fall 2002 # 8. For each continuous function f on the interval $[0, 1]$, let

$$(Tf)(x) = x + \lambda \int_0^x f(t) \sin \pi t \, dt.$$

a. Find a range of values of the parameter λ for which the transformation T is a contraction with respect to the supremum norm. Justify your answer.

b. Find a range of values of the parameter λ for which the transformation T is a contraction with respect to the L^2 norm. Justify your answer.

c. Describe the iterative process for solving the integral equation

$$f(x) = x + \lambda \int_0^x f(t) \sin \pi t \, dt$$

specifying the transformation to be iterated and explaining how an why this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the iterates $f_1(x)$ and $f_2(x)$.

End of Exam