

California State University – Los Angeles

Mathematics

Masters Degree Comprehensive Examination

Linear Analysis Fall 2016
Gutarts, Hoffman, Krebs*

Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

- (1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
- (2) Write on one side of the paper only
- (3) Begin each problem on a new page
- (4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b)$$

$$2 \cos a \cos b = \cos(a - b) + \cos(a + b)$$

$$2 \sin a \cos b = \sin(a + b) + \sin(a - b)$$

$$2 \cos a \sin b = \sin(a + b) - \sin(a - b)$$

$$\int \sin^2(ax) dx = \frac{x}{2} - \frac{1}{4a} \sin(2ax)$$

$$\int \cos^2(ax) dx = \frac{x}{2} + \frac{1}{4a} \sin(2ax)$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$$

Fall 2016 # 1. For each of the following sets of sequences with entries in \mathbb{R} determine whether it is a vector space over \mathbb{R} . Give reasons for your answers. You may assume the the set of all sequences in \mathbb{R} with the operations

$$(a_1, a_2, a_3, \dots) + (b_1, b_2, b_3, \dots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots)$$

$$\lambda(a_1, a_2, a_3, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \dots)$$

is a vector space over \mathbb{R} .

- a.** $A = \{(a_1, a_2, \dots) : a_1 = 1\}$ **b.** $B = \{(a_1, a_2, \dots) : a_2 = a_3\}$
c. $C = \{(a_1, a_2, \dots) : |a_k| \leq 2 \text{ for each } k\}$

Fall 2016 # 2. For each of the following, decide if the formula given defines a norm on \mathbb{R}^2 . If it does, prove it. If it does not, explain why not. (One is a norm and one is not.)

- a.** $\|(a, b)\|_\alpha = 5|a| - 7|b|$ **b.** $\|(a, b)\|_\beta = \int_{-1}^1 |at + b| dt$

Fall 2016 # 3. Suppose \mathcal{X} and \mathcal{Y} are normed spaces and $T : \mathcal{X} \rightarrow \mathcal{Y}$ is a bounded linear transformation from \mathcal{X} into \mathcal{Y} .

- a.** Show that the kernel (null space) of T is a vector subspace of \mathcal{X} .
b. Show that the subspace $\ker(T)$ of part **(a)** is a closed subset of \mathcal{X} .

Fall 2016 # 4. Let the linear operator $K : C([0, 1]) \rightarrow C([0, 1])$ be defined for each continuous real valued function f on $[0, 1]$ by $(Kf)(x) = \lambda \int_0^x tf(t) dt$.

- a.** Find a range of values for the parameter λ for which the operator norm of K is strictly less than 1 with respect to the norm on $C([0, 1])$ given by $\|g\|_\infty = \sup\{|g(t)| : t \in [0, 1]\}$.
b. Describe an iterative process for solving (generating approximate solutions to) the integral equation

$$f(x) = 1 + \lambda \int_0^x tf(t) dt \quad \text{for all } x \text{ in } [0, 1]$$

specifying the transformation to be iterated and explaining what the solution to part a has to do with the convergence of your method to a solution.

- c.** With $f_1(x) = 1$ for all x , compute the next two iterations $f_2(x)$ and $f_3(x)$.

Fall 2016 # 5. Let $K : C([0, 1]) \rightarrow C([0, 1])$ be defined for each continuous function f and $[0, 1]$ by

$$(Kf)(x) = \int_0^1 (x^2t + t^2x)f(t) dt$$

- a.** Find the dimension of the range of K .
b. Find all non-zero eigenvalues of K .

Fall 2016 # 6. Here are definitions for modes of convergence of a sequence of bounded linear operators on a Hilbert space \mathcal{H} .

Operator norm convergence: $T_n \xrightarrow{\|\cdot\|} T$ if $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$.

Strong operator convergence: $T_n \xrightarrow{\text{sop}} T$ if $T_n f \rightarrow T f$ in \mathcal{H} as $n \rightarrow \infty$ for each f in \mathcal{H} .

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for \mathcal{H} and P_n be the orthogonal projection onto the closed linear span of $\{e_1, \dots, e_n\}$.

- Show that if $T_n \xrightarrow{\|\cdot\|} T$, then $T_n \xrightarrow{\text{sop}} T$.
- Show that $P_n \xrightarrow{\text{sop}} I$. (I is the identity operator on \mathcal{H} .)
- Show that the P_n cannot converge in operator norm to anything. (Suggestion: What can you say about $\|P_{n+1} - P_n\|$?)

Fall 2016 # 7. If $T : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded linear operator from a Hilbert space \mathcal{H} over \mathbb{C} to itself, then the *numerical range* of T is defined as

$$W(T) = \{\langle Tf, f \rangle \in \mathbb{C} : f \in \mathcal{H} \text{ and } \|f\| = 1\}$$

- Show that $W(T) \subseteq \{\mu \in \mathbb{C} : |\mu| \leq \|T\|\}$
- Show that if μ is an eigenvalue for T , then $\mu \in W(T)$.

Fall 2016 # 8. Suppose $f(x)$ is a 2π -periodic real valued function on \mathbb{R} and that f'' exists and is continuous on \mathbb{R} .

- Show that for $k \neq 0$, $\widehat{f}(k) = -\frac{1}{k^2} \widehat{f''}(k)$
- Show the Fourier series for f is absolutely convergent.

End of Exam