California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Spring 2004

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Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}$ denotes the set of real numbers.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

$\bar{z}$ denotes the complex conjugate of the complex number z.

$|z|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\,dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a\cos b + \cos a \sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

$$\int \ln x\,dx = x\ln x - x$$

Spring 2004 # 1. Let a be a real constant with $0 < a < \pi$. Define f on $[-\pi, \pi]$ by

$$f(t) = \begin{cases} 1, & \text{for } |t| \le a \\ 0, & \text{for } a < |t| \le \pi \end{cases}.$$

a. Compute either the exponential or the trigonometric form of the Fourier series for f on $[-\pi, \pi]$. (Your choice which)

b. Use the result of part **a** to show that $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2(ka)}{k^2} = \frac{a(\pi - a)}{2}$

Spring 2004 # 2. Let $\mathcal{V}$ be the space $\mathcal{C}([0,2], \mathbb{R})$ of all real valued continuous functions on $[0,2]$ equipped with the inner product $\langle f, g \rangle = \int_0^2 f(t)g(t)\,dt$.

Let $\mathcal{W}$ be the subspace of $\mathcal{V}$ spanned by the functions $f_1(x) = 1$ and $f_2(x) = x$.

a. Prove that f_1 and f_2 are linearly independent. (In the process, state clearly what it means for f_1 and f_2 to be linearly independent as functions on $[0,2]$.)

b. Find a basis for $\mathcal{W}$ orthonormal with respect to the specified inner product.

c. Find constants a and b which minimize the quantity $\int_0^2 (a + bx - 2x^2)^2\,dt$.

Spring 2004 # 3. For f in the space $C([-\pi, \pi])$ of all continuous numerical valued functions on $[-\pi, \pi]$, let

$$\phi(f) = \int_{-\pi}^{\pi} f(t)\,dt$$

a. Show that ϕ is a linear functional on $[-\pi, \pi]$.

b. Show that ϕ is continuous when the L^1 norm, $\| f \|_1 = \int_{-\pi}^{\pi} |f(t)|\,dt$, is used on $C([-\pi, \pi])$.

c. Show that ϕ is continuous when the L^2 norm, $\| f \|_2 = \int_{-\pi}^{\pi} |f(t)|^2\,dt$, is used on $C([-\pi, \pi])$.

d. Show that ϕ is continuous when the uniform norm, $\| f \|_\infty = \sup\{|f(t)| : t \in [-\pi, \pi]\}$, is used on $C([-\pi, \pi])$.

Spring 2004 # 4. Suppose $\mathcal{H}$ is an inner product space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. A sequence $\{f_n\}_{n=1}^{\infty}$ is $\mathcal{H}$ is said to converge weakly to a weak limit g in $\mathcal{H}$ (written $f_n \xrightarrow{w} g$) if $\langle f_n, h \rangle \to \langle g, h \rangle$ as numbers for every h in $\mathcal{H}$.

a. Show that if $\| f_n - g \| \to 0$ as $n \to \infty$, then $f_n \xrightarrow{w} g$.
(Norm convergence implies weak convergence.)

b. Suppose $e_1, e_2, e_3, \ldots$ is an infinite orthonormal sequence in $\mathcal{H}$. Show that $e_n \xrightarrow{w} 0$

c. Use part **b** to show that in an infinite dimensional inner product space weak convergence does not imply norm convergence

d. Show that in a finite dimensional inner product space, weak convergence does imply norm convergence.

Spring 2004 # 5. For complex valued functions f on $[-\pi, \pi]$, define Kf by

$$(Kf)(x) = \int_{-\pi}^{\pi} (\cos t + x \sin t) f(t)\, dt.$$

a. Describe the kernel and range of K.
b. Find any nonzero eigenvalues in $\mathbb{C}$ and corresponding eigenspaces.
c. Find a function f on $[-\pi, \pi]$, such that

$$f(x) = x^2 + \frac{1}{\pi} \int_{-\pi}^{\pi} (\cos t + x \sin t) f(t)\, dt.$$

(You may use the facts that $\int_{-\pi}^{\pi} t \sin t\, dt = 2\pi$ and $\int_{-\pi}^{\pi} t^2 \cos t\, dt = -4\pi$.)

Spring 2004 # 6. Supppose g is a continuous function on $[0, 1]$. For f in $C([0, 1])$ define Tf by

$$(Tf)(x) = g(x) + \lambda \int_0^1 e^{x-t} f(t)\, dt.$$

a. Find a range of values of λ for which T is a contraction with respect to the supremum norm on $C([0, 1])$.
b. Find a range of values of λ for which T is a contraction with respect to the L^2-norm on $C([0, 1])$.
c. Describe the iterative process for finding a solution f to the equation

$$f(x) = g(x) + \lambda \int_0^1 e^{x-t} f(t)\, dt$$

explaining how the procedure works and how one knows that it leads to a solution.
d. With $f_0(x) = 0$ for all x, compute the first three iterates, f_1, f_2, and f_3.

Spring 2004 # 7. Consider the boundary value problem

$$(*) \qquad -\frac{d}{dx}[x f'(x)] = \phi(x) \quad \text{for} \quad 1 \le x \le 2 \qquad \text{with} \quad f(1) = 0 \text{ and } f'(2) = 0$$

a. Find a function $G(x, t)$ such that solutions f to the boundary value problem $(*)$ for known functions ϕ are given by

$$f(x) = \int_1^2 G(x, t) f(t)\, dt$$

b. Solve the boundary value problem $(*)$ for $f(x)$ when $\phi(x) = 1$ for all x.

End of Exam