California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Linear Analysis Spring 2005
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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a, b])$ denotes the space of all continuous functions on the interval $[a, b]$. If there is need to specify the possible values, $\mathcal{C}([a, b], \mathbb{R})$ will denote the space of all continuous real valued functions on $[a, b]$ and $\mathcal{C}([a, b], \mathbb{C})$ the space of all continuous complex valued functions.
$L^2([a, b])$ denotes the space of all functions on the inteval $[a, b]$ such that $\int_a^b |f(x)|^2 \, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\, dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Spring 2005 # 1. Let $f(x) = |x|$ for x in $[-\pi, \pi]$.

a. Compute either the exponential or the trigonometric form of the Fourier series for f on $[-\pi, \pi]$. (Your choice which)

b. Use the result of part **a** to show that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}$

Spring 2005 # 2. Let $\mathcal{Y}$ be the space $C([0,1])$ of all continuous real valued functions on $[0,1]$.

Let $\mathcal{Y}$ be the space of all f in $C([0,1])$ such that f' exists and is continuous on $[0,1]$ (with appropriate one-sided limits at the ends).

For f in $\mathcal{X}$, let $Df = f'$ and $\| f \|_s = \| f \|_\infty + \| f' \|_\infty$.

a. Show that $\| \cdot \|_s$ is a norm on $\mathcal{X}$. (You may use, if you wish, facts about the known supremum norm $\| \cdot \|_\infty$ on $C([0,1])$.)

b. Define what it means to be a linear operator and show that D is a linear operator from $\mathcal{X}$ into $\mathcal{Y}$.

c. Show that D is not continuous if the norm $\| \cdot \|_\infty$ is used on both $\mathcal{X}$ and $\mathcal{Y}$.

d. Show that D is continuous if the norm $\| \cdot \|_s$ is used on $\mathcal{X}$ and $\| \cdot \|_\infty$ is used on $\mathcal{Y}$.

e. Find the operator norm of D in the setting of part **(d)**.

Suggestion: In parts **c**, **d**, and **e** the functions $f_n(t) = (1/n)\sin(nt)$ might be useful.

Spring 2005 # 3. For f and g in the space $C([0,1])$ of all continuous complex valued functions on $[0,1]$, let

$$[f, g] = \int_0^1 f(t)\overline{g(t)}\, t\, dt$$

Let $\mathcal{P}^1$ be the subspace of $\mathcal{C}([0,1])$ consisting of all polynomials of degree no more than 1.

a. Show that $[\cdot, \cdot]$ is an inner product on $C([0,1])$. (You may use, if you wish, facts about the known inner product $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt$ on $C([0,1])$.)

b. Find polynomials $e_0(t)$ and $e_1(t)$ which form a basis for $\mathcal{P}^1$ which is orthonormal with respect to the inner product $[\cdot, \cdot]$.

c. Use the result of part **(b)** to find numbers a and b which make the quantity $J = \int_0^1 \left| t^3 - a - bt \right|^2 t\, dt$ as small as possible.

Spring 2005 # 4. Let $\mathcal{X}$ be the space of all continuous 2π-periodic functions on $\mathbb{R}$ with the inner product $\langle f, g \rangle = (1/\pi) \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$ and its associated norm. For each positive integer k and each f in $\mathcal{X}$, let $b_k(f) = \int_{-\pi}^{\pi} f(t) \sin kt\, dt$.

a. Show that b_k is a continuous linear functional on $\mathcal{X}$.

b. Find the norm of b_k as a linear functional on $\mathcal{X}$.

c. Show that $\lim_{k \to \infty} b_k(f) = 0$ for each f in $\mathcal{X}$.

Spring 2005 # 5. For each continuous function f on the interval $[0,1]$, define a function Tf by

$$(Tf)(x) = 1 + \lambda \int_0^x t f(t)\, dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm. Justify your answer.

b. Describe the iterative process for solving the integral equation

$$f(x) = 1 + \lambda \int_0^x t f(t)\, dt \qquad (*)$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 1$ for all x as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

c. Explain why the equation $(*)$ is equivalent to the initial value problem

$$f'(x) = \lambda x f(x) \quad ; f(0) = 1 \qquad (**)$$

Spring 2005 # 6. **a.** Suppose S is a bounded linear operator on a Banach space $\mathcal{X}$. Describe the Neumann series method for finding $(I - S)^{-1}$ and state sufficient conditions for convergence of the series.

b. Let T be the operator on $\mathbb{R}^3$ given with respect to the standard basis by the matrix $T = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$. Use the Neumann series to find T^{-1}. (Hint: What do you want to use as S ?)

Spring 2005 # 7. Suppose T is a bounded linear operator on a Hilbert space $\mathcal{H}$.

a. Define what it means for a number μ to be an eigenvalue for T and what it means for μ to be in the spectrum of T.

b. Show that if T is self-adjoint and μ is an eigenvalue for T, then μ is a real number.

c. Give an example of an operator T and a number μ which is in the spectrum of T but is not an eigenvalue for T. Justify your answer.

End of Exam