California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Spring 2007

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Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|\, z \,|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |\, f(x) \,|^2 \; dx < \infty$

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\qquad$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$ $\qquad$ $2 \cos a \cos b = \cos(a-b) + \cos(a+b)$

$2 \sin a \cos b = \sin(a+b) + \sin(a-b)$ $\qquad$ $2 \cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} \qquad \int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2}$$

Spring 2007 # 1. Let $f(x) = \begin{cases} 0, & \text{for } -\pi \le x < -\pi/2 \\ 1, & \text{for } -\pi/2 \le x \le \pi/2 \\ 0, & \text{for } \pi/2 < x < \pi \end{cases}$
and consider f to be extended to a 2π periodic function on $\mathbb{R}$.

a. Find the Fourier series for the function f on the interval $[-\pi, \pi]$. (You may use either the trigonometric or exponential form.)

b. Use the result of part **a** to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

Spring 2007 # 2. Assume as known that the formula $[f, g] = \int_0^1 f(t)g(t)\,t\,dt$ gives an inner product on the space of all continuous real valued functions on the interval $[0, 1]$. Let $\mathcal{P}_1$ be the subspace consisting of all polynomials with real coefficients with degree no more than 1.

a. Find a basis for $\mathcal{P}_1$ which is orthonormal with respect to the inner product $[\cdot, \cdot]$.

b. Use the result of part **a** to find constants a and b making the quantity

$$J = \int_0^1 (ax + b - e^x)^2 x\,dx$$

as small as possible.

(Note: there are other ways to do part **b**, but you are asked to use part **a** to illustrate a method.)

(You may use the facts that $\int_0^1 te^t\,dt = 1$ and $\int_0^1 t^2 e^t = e - 2$.)

Spring 2007 # 3. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator from a Hilbert space $\mathcal{H}$ into itself.

Let $\text{range}(T) = \{Tx : x \in \mathcal{H}\}$.

Let $\ker(T) = \{x \in \mathcal{H} : Tx = 0\}$

a. Show that $\text{range}(T)$ and $\ker(T)$ are vector subspaces of H.

b. Show that $\ker(T)$ is a closed subset of H.

c. Give a definition of the orthogonal complement, $A^\perp$, of a subset A of H.

d. Show that if $A \subseteq \mathcal{H}$, then $A^\perp$ is a vector subspace of $\mathcal{H}$.

Spring 2007 # 4. For each of the following decide whether the suggetsed formula defines a norm on the indicated space. Give reasons for your answers. (You may assume that $\|f\|_1 = \int_0^1 |f(t)|\,dt$ does give a norm on the space of all continuous functions on the interval $[0, 1]$.

a. $\mathcal{V}_a = \mathbb{R}^2$ $\quad \|(x, y)\|_a = |x + y|$

b. $\mathcal{V}_b = \mathbb{R}^2$ $\quad \|(x, y)\|_b = \max(|x|, |y|)$

c. $\mathcal{V}_c = \mathbb{R}^2$ $\quad \|(x, y)\|_c = \int_0^1 |x + yt|\,dt$

d. $\mathcal{V}_d$ is the space of all differentiable functions f on $[0, 1]$ with f' continuous. $\|f\|_d = \int_0^1 |f'(t)|\,dt$

Spring 2007 # 5. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Suppose $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator such that $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all f and g in $\mathcal{H}$.

a. Show that all eigenvalues of T are real.

b. Show that eigenvectors of T corresponding to different eigenvalues are orthogonal with respect to the innerproduct $\langle \cdot, \cdot \rangle$

(Do not just quote a theorem you know. Prove it.)

Spring 2007 # 6. For each continuous function f on the interval $[0, 1]$, define Tf on $[0, 1]$ by

$$(Tf)(x) = e^x + \lambda \int_0^x e^{x-t} f(t)dt$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm $\| f \|_\infty = \sup_{x\in[0,1]} | f(x) |$. Justify your answer.

b. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the L^2 norm $\| f \|_2^2 = \int_0^1 | f(x) |^2 \, dx$. Justify your answer.

c. Describe the iterative process for solving the integral equation

$$f(x) = e^x + \lambda \int_0^x e^{x-t} f(t)dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

Spring 2007 # 7. **a.** Suppose $S : \mathcal{X} \to \mathcal{X}$ is a bounded linear operator on a complete normed space $\mathcal{X}$. Explain the Neumann series method for finding $(I - S)^{-1}$, and give a condition on S (other than $S = 0$) sufficient to justify its use. (Proof not required.)

b. Use the method of part **a** to compute $\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/3 & 1 \end{pmatrix}^{-1}$.

Spring 2007 # 8. For each continuous function f on the interval $[0, 1]$, let the function Kf be defined by

$$(Kf)(x) = \int_0^1 (3xt - 1)f(t)\, dt.$$

a. Find all nonzero numbers μ such for which there are nonzero continuous functions f with $Kf = \mu f$. For each of these μ find the corresponding solutions f.

b. Find a continuous function g on $[0, 1]$ such that

$$g(x) = 1 + \int_0^1 (3xt - 1)g(t)\, dt.$$

End of Exam