CSULA Mathematics
Masters Degree Comprehensive Examination
Linear Analysis Spring 2018
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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Spring 2018 # 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \pi - x$ on the interval $[0, \pi]$ then extended to be 2π-periodic and even on $\mathbb{R}$.

a. Find the Fourier series for f.

b. Use the result of part **(a)** to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$

Spring 2018 # 2. Let $\mathcal{X} = C([0, 1])$ be the space of continuous real valued functions on $[0, 1]$ with the two norms

$$\| f \|_1 = \int_0^1 | f(t) | \, dt \qquad \text{and} \qquad \| f \|_\infty = \sup\{| f(t) | : t \in [0, 1]\}.$$

a. Pick one of these two (your choice) and show that it is a norm on $\mathcal{X}$.
b. Show that $\| f \|_1 \le \| f \|_\infty$ for every f in $\mathcal{X}$.
c. Are these two norms equivalent on $\mathcal{X}$? (Prove or disprove.)

Spring 2018 # 3. Let ℓ^2 be the space of complex valued sequences $\vec{x} = (x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} | x_n |^2 < \infty$ with the usual norm and inner product. For $\vec{x}$ in ℓ^2, let

$$A(\vec{x}) = (0, x_1, x_2, x_3, \dots) \qquad \text{and} \qquad B(\vec{x}) = (x_2, x_3, x_4, \dots)$$

a. Show that A and B are linear from ℓ^2 into ℓ^2.
b. Show that A and B are bounded as linear operators on ℓ^2.
c. Find the operator norms $\| A \|$ and $\| B \|$.
d. Find AB and BA.

Spring 2018 # 4. Let ℓ^2 be the space of complex valued sequences $\vec{x} = (x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} |\, x_n \,|^2 < \infty$ and norm given by $\|\, \vec{x} \,\|^2 = \sum_{n=1}^{\infty} |\, x_n \,|^2$. For each of the following subsets of ℓ^2 decide whether it is a vector subspace of ℓ^2. If it is, prove it. If not, explain how you know it is not.

a. $A = \{\vec{x} \in \ell^2 : \|\, \vec{x} \,\| \leq 1\}$
b. $B = \{\vec{x} \in \ell^2 : x_1 + x_2 = 0\}$
c. $C = \{\vec{x} \in \ell^2 : x_1 = \sum_{n=2}^{\infty} x_n^2\}$

Spring 2018 # 5. For each of the subsets of ℓ^2 in the previous problem:

a. Determine whether it is a closed subset of ℓ^2.
b. Determine whether it is a convex subset of ℓ^2.

Spring 2018 # 6. Let $\mathcal{X} = C([0,1])$ be the space of continuous complex valued functions on $[0,1]$. Let a be a real constant with $0 < a < 1$. Define $\phi : \mathcal{X} \to \mathbb{C}$ by:

$$\phi(f) = \int_0^a x^2 f(x)\, dx$$

a. Show $\langle\, f, g \,\rangle = \int_0^1 f(x)\overline{g(x)}\, dx$ is an inner product on $\mathcal{X}$
b. Show that ϕ is linear.
c. Show that ϕ is continuous with respect to the norm associated to the inner product of part **(a)**.

Spring 2018 # 7. For f in the space $C([0,2])$ of continuous functions on the interval $[0,2]$ and each number λ, let Tf be defined on $[0,2]$ by

$$(Tf)(x) = x + \lambda \int_0^x (x-t) f(t)\, dt$$

a. Show that f is a solution to the integral equation

$$(VIE) \qquad f(x) = x + \lambda \int_0^x (x-t) f(t)\, dt$$

if and only if it is a solution to the initial value problem

$$(IVP) \qquad f''(x) = \lambda f(x) \quad \text{with } f(0) = 0 \text{ and } f'(0) = 1.$$

b. Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0,2])$ with respect to the supremum norm $\|\, f \,\|_\infty = \sup_{x \in [0,2]} |\, f(x) \,|$. Justify your answer.
c. Describe the iterative process for solving the integral equation (VIE) of part **(a)** specifying the transformation to be iterated and explaining why this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the iterates, $f_1(x)$and $f_2(x)$.

End of Exam