CSULA Mathematics
Masters Degree Comprehensive Examination
Linear Analysis Spring 2020
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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Spring 2020 # 1. For each continuous function f on the interval $[0, 2]$ define a function Tf by

$$(Tf)(x) = x + \lambda \int_0^x xtf(t)\, dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0, 2])$ with respect to the supremum norm, $\| f \|_\infty = \sup\{| f(t) | : t \in [0, 2]\}$. Justify your answer.

b. Describe the iterative process for solving the integral equation

$$f(x) = x + \lambda \int_0^x xtf(t)\, dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first two iterates, $f_1(x)$, and $f_2(x)$.

c. Show that if f is a solution to the integral equation of part (b), then it is also a solution to the differential equation

$$f''(x) - \lambda x^2 f'(x) - 3\lambda x f(x) = 0 \qquad \text{with } f(0) = 0 \text{ and } f'(0) = 1.$$

Spring 2020 # 2. Let (x, y) and (a, b) represent points in $\mathbb{R}^2$.

a. For each of the following decide whether the formula given for $\| (x, y) \|$ defines a norm on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.

(i) $\| (x, y) \|_{(i)} = \sqrt{x^4 + y^4}$
(ii) $\| (x, y) \|_{(ii)} = | x | + 5 | y |$

b. For each of the following decide whether the formula given for $\langle (a, b), (x, y) \rangle$ defines an inner product on $\mathbb{R}^2$. If it does, prove it. If it does not, explain how you know it does not.

(i) $\langle (a, b), (x, y) \rangle_{(i)} = 2ax + 3by$
(ii) $\langle (a, b), (x, y) \rangle_{(ii)} = ax^2 - by^2$

Spring 2020 # 3. Let $\mathcal{P}^1$ be the space of all polynomials of degree no more than 1 with the inner product $\langle f, g \rangle = \int_0^2 f(t)\overline{g(t)}\, dt$

a. Find a basis for $\mathcal{P}^1$ which is orthonormal with respect to that inner product

b Find constants a and b which make the quantity $J = \int_0^2 \left| t^3 - a - bt \right|^2 dt$ as small as possible.

Spring 2020 # 4. For each of the following, determine if it is a vector space over $\mathbb{R}$. Give reasons for your answers.

- **a.** The set A of integrable real valued functions f on $[0,1]$ with $\int_0^1 f(t)\, dt = 0$.
- **b.** The set B of all polynomials with real coefficients and even degree.
- **c.** The set C of differentiable real valued functions on $\mathbb{R}$ with $f(0) + f'(0) = 0$ and $f(1) - f'(1) = 1$.
- **d.** The set D of all 2×2 matrices with real entries and determinant 0.

Spring 2020 # 5. Let $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < \pi \\ -1, & \text{for } -\pi < x < 0 \\ 0 & \text{for } x = -\pi, 0, \pi \end{cases}$$

and extending 2π-periodically.

- **a.** Find the Fourier series for $f(x)$. (Exponential form or trigonometric form, your choice)
- **b.** Use the result of part **a** to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

Spring 2020 # 6. Let $C([0,1],\mathbb{R})$ be the space of all continuous real valued functions on the interval $[0,1]$ with the norm $\| f \|_\infty = \sup\{| f(t) | : t \in [0,1]\}$.

Define $\phi : C([0,1],\mathbb{R}) \to \mathbb{R}$ by $\phi(f) = f(0) + \int_0^1 f(t)\, dt$

- **a.** Show ϕ is linear.
- **b.** Show ϕ is continuous.
- **c.** Find the operator norm of ϕ. Justify your answer.

Spring 2020 # 7. **a.** Describe the Neumann series method for inversion of a bounded linear operator of the form $I - S$ on a Banach space $\mathcal{X}$. Be sure to include conditions sufficient to ensure convergence of the method.

b. Let B be the linear operator on $\mathbb{R}^3$ represented with respect to the standard orthonormal basis by the matrix $\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$. Find the operater norm of B.

c. Let T be the linear operator on $\mathbb{R}^3$ represented with respect to the standard orthonormal basis by the matrix $\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}$. Use the method of part **a** to find (the matrix for) T^{-1}. For full credit, actually find the 3×3 matrix, do not just leave it as a series.

End of Exam