California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Winter 2002

Hoffman*, Meyer, Verona

Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\, dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Winter 2002 # 1. Let $\mathcal{X}$ be the space of continuous functions on $[-\pi, \pi]$ with the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$ and the associated norm. For f in $\mathcal{X}$, put $\phi_n(f) = \int_{-\pi}^{\pi} f(t) \cos nt\, dt$

a. Show that ϕ_n is a linear functional on $\mathcal{X}$.
b. Show that ϕ_n is bounded as a linear functional on $\mathcal{X}$
c. Find $\| \phi_n \|$
d. Show that $\lim_{n\to\infty} \phi_n(f) = 0$ for each f in $\mathcal{X}$.

Winter 2002 # 2. Suppose $\mathcal{E} = \{e_1, e_2, e_3, \dots\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$. Let $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ be numbers. For v in $\mathcal{H}$, put $T_n v = \sum_{k=1}^{n} \lambda_k \langle v, e_k \rangle e_{k+1}$

a. Show that T_n is a linear operator from $\mathcal{H}$ into $\mathcal{H}$.
b Show that T_n is a bounded linear operator.
c. Show that the operator norm of T_n is $\| T_n \| = \max(\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\})$.
d. Describe the matrix for T_n with respect to the orthonormal basis $\mathcal{E}$.

Winter 2002 # 3. Suppose $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are norms on a vector space $\mathcal{X}$. For each of the following decide whether the proposed formula defines a norm on $\mathcal{X}$. If it does, prove it. If not, explain how you know why not.

a. $\| v \|_a = \| v \|_\alpha^2 + \| v \|_\beta^2$
b. $\| v \|_b = 3 \| v \|_\alpha + 2 \| v \|_\beta$
c. $\| v \|_c = \| v \|_\alpha \cdot \| v \|_\beta$

Winter 2002 # 4. For continuous functions f on the interval $[0, 1]$, let

$$(Kf)(x) = \int_0^\pi f(t) \cos x \cos t\, dt$$

a. Find all nozero constants μ for which there are nonzero solutions ϕ to the equation $(K\phi)(x) = \mu\phi(x)$. State the value or values μ and the corresponding functions ϕ.
b. Find a function $R(x, t; \lambda)$ such that solutions to the integral equation

$$\text{(IE)} \qquad f(x) = g(x) + \lambda \int_0^\pi f(t) \cos x \cos t\, dt$$

are given by

$$f(x) = g(x) + \lambda \int_0^\pi R(x, t; \lambda) g(t)\, dt$$

c. Solve the integral equation $f(x) = x + \dfrac{1}{\pi} \displaystyle\int_0^\pi f(t) \cos x \cos t\, dt.$

(You may use any method of your choice, the result of part **b** is one possibility.)

Winter 2002 # 5. Let a be a real constant with $0 < a < \pi$.
Put $f(x) = 1$ for $|x| \le a$ and $f(x) = 0$ for $a < |x| \le \pi$.

a. Compute the Fourier series for f on $[-\pi, \pi]$. (Trigonometric or exponential, your choice)

b. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \sin^2 ka = \frac{a(\pi - a)}{2}$.

Winter 2002 # 6. Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis vectors for $\mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation represented with respect to the standard basis by the matrix $M = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

(a) Find all the eigenvalues of T

(b) Find an orthonormal basis for $\mathbb{R}^2$ which consists of eigenvectors of T.

(c) Show that if $\vec{v} \in \mathbb{R}^2$, then

$$\lim_{n\to\infty} T^n \vec{v} = \vec{0}.$$

Winter 2002 # 7. Suppose $\mathcal{H}$ is a Hilbert space and T and T_n, $n = 1, 2, 3, \ldots$ are bounded linear operators on $\mathcal{H}$. We know that the sequence T_n is said to converge to T in operator norm and write $T_n \xrightarrow{\|\cdot\|} T$ if $\lim_{n\to\infty} \| T_n - T \| = 0$. Here are two other possible notions of convergence for a sequence of Hilbert space operators:

Strong Operator Convergence: The T_n are said to converge **strongly** to T and we write $T_n \xrightarrow{s} T$ if $T_n f \to Tf$ in norm for every f in $\mathcal{H}$.

Weak Operator Convergence: The T_n are said to converge **weakly** to T and we write $T_n \xrightarrow{w} T$ if $\langle T_n f, g \rangle \to \langle Tf, g \rangle$ in $\mathbb{F}$ for every $f$and g in $\mathcal{H}$.

a. Show that if $T_n \xrightarrow{\|\cdot\|} T$, then $T_n \xrightarrow{s} T$. (Operator norm convergence implies strong operator convergence .)

b. Show that if $T_n \xrightarrow{s} T$, then $T_n \xrightarrow{w} T$. (Strong operator convergence implies weak operator convergence .)

c. Suppose $\mathcal{E} = \{e_1, e_2, e_3, \ldots\}$ is an orthonormal basis for $\mathcal{H}$. Let P_n be the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{M}_n = \text{span}(\{e_1, e_2, \ldots, e_n\})$ and I be the identity operator on $\mathcal{H}$. ($If = f$ for every f in $\mathcal{H}$). Show that $P_n \xrightarrow{s} I$.

d. Use the result of part **(d)** to show that strong operator convergence does not imply operator norm convergence. $\left(T_n \xrightarrow{s} T\right) \not\Rightarrow \left(T_n \xrightarrow{\|\cdot\|} T\right)$.

Winter 2002 # 8. Let ϕ be a continuous real valued function on $0 \le x \le \pi$ and consider the boundary value problem

$$(**) \qquad f''(x) + f(x) = \phi(x) \quad \text{for } 0 \le x \le \pi \quad \text{with } f(0) + f'(0) = 0 \text{ and } f(\pi) = 0$$

a. Find a function $G(x,t)$ such that the solutions to (**) are given by

$$f(x) = \int_0^\pi G(x,t)\phi(t)\,dt$$

b. Solve (**) with the function $\phi(x) = 1$. You may use your result from part **(a)** or any other method you wish.

End of Exam