Mon
2/10
week 4

Recall

$$\sigma(n) = \sum_{d \mid n} d$$

$$\boxed{\sigma(6) = \underbrace{1+2+3}_{6} + \underbrace{6}_{6} = \underbrace{12}_{2n}}$$

(n=6)

$$\sigma(8) = 1+2+4+8 = 15$$

if

From last time
(but with proof filled in)

$\boxed{\text{Fact 33}}$

n is perfect
iff $\sigma(n) = 2n$.

proof:

n is perfect

iff $\sum_{\substack{d|n \\ d \neq n}} d = n$

add n to both sides

iff $\sum_{d|n} d = 2n$

iff $\sigma(n) = 2n$.

$\blacksquare$

Last time

$\sigma(12) = \sigma(2^2 \cdot 3^1)$

$= \sigma(2^2)\,\sigma(3^1)$

we calculated this last time

New stuff

Theorem 35

(1) If p is prime and e is a positive integer then $\sigma(p^e) = \dfrac{p^{e+1}-1}{p-1}$

(2) If $p_1, p_2, \ldots, p_k$ are distinct primes and $e_1, e_2, \ldots, e_k$ are positive integers then

$$\sigma\left(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}\right) = \sigma\left(p_1^{e_1}\right)\sigma\left(p_2^{e_2}\right)\cdots\sigma\left(p_k^{e_k}\right) = \left(\frac{p_1^{e_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{e_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_k^{e_k+1}-1}{p_k-1}\right)$$

(3) If a and b are positive integers with $\gcd(a,b)=1$ then $\sigma(ab)=\sigma(a)\sigma(b)$

Geometric sum

If $x \neq 1$, then

$1+x+x^2+\cdots+x^n = $

proof:

① Since p is prime,

$$\sigma(p^e) = 1 + p + p^2 + \dots + p^e$$
$$= \frac{p^{e+1}-1}{p-1}.$$

Geometric sum

If $x \neq 1$, then

$$1+x+x^2+\dots+x^n = \frac{x^{n+1}-1}{x-1}$$

pf: $(1+x+x^2+\dots+x^n)(x-1)$

$$= x+x^2+x^3+\dots+x^n+x^{n+1} -1-x-x^2-x^3-\dots-x^n$$
$$= -1+x^{n+1}. \quad \boxed{//}$$

side example

$$\sigma(5^3) = 1+5+5^2+5^3$$

② We induct on k.

By part 1, if $k=1$ then $\sigma(p_1^{e_1}) = \frac{p_1^{e_1+1}-1}{p_1-1}$.

Let $p_1, p_2, \dots, p_k, p_{k+1}$ be distinct primes and $e_1, e_2, \dots, e_k, e_{k+1}$ be positive integers, where $k \geq 1$.

Assume that

$$\sigma(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}) = \sigma(p_1^{e_1})\,\sigma(p_2^{e_2}) \cdots \sigma(p_k^{e_k}).$$

Using this we get

$$\sigma(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} p_{k+1}^{e_{k+1}}) = \sum_{d \mid p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} p_{k+1}^{e_{k+1}}} d = \sum_{\substack{d = p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k} p_{k+1}^{f_{k+1}} \\ 0 \le f_i \le e_i}} p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k} p_{k+1}^{f_{k+1}}.$$

Side example

$$\sum_{d \mid 2^1 5^3} d = 2^0 5^0 + 2^0 5^1 + 2^0 5^2 + 2^0 5^3 + 2^1 5^0 + 2^1 5^1 + 2^1 5^2 + 2^1 5^3 = \sum_{\substack{d = 2^{f_1} 5^{f_2} \\ 0 \le f_1 \le 1 \\ 0 \le f_2 \le 3}} 2^{f_1} 5^{f_2}$$

Let $S = \sum_{0 \le f_i \le e_i} p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k} = \sigma(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})$

Th

$\sigma(p$

-1

$(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})$

Then,

$$\begin{aligned}
\sigma(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} p_{k+1}^{e_{k+1}}) &= S p_{k+1}^{0} + S p_{k+1}^{1} + S p_{k+1}^{2} + \cdots + S p_{k+1}^{e_{k+1}} \\
&= S(1 + p_{k+1}^{1} + p_{k+1}^{2} + \cdots + p_{k+1}^{e_{k+1}}) \\
&= S \cdot \sigma(p_{k+1}^{e_{k+1}}) \\
&= \sigma(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}) \cdot \sigma(p_{k+1}^{e_{k+1}}) \\
&\overset{\boxed{\text{assumption}}}{=} \sigma(p_1^{e_1})\sigma(p_2^{e_2})\cdots\sigma(p_k^{e_k})\sigma(p_{k+1}^{e_{k+1}}).
\end{aligned}$$

By induction this is true for all $k \geq 1$.

(3) Let a, b be positive integers with $a \geq 2$ and $b \geq 2$.

Suppose $\gcd(a,b) = 1$.

Since $\gcd(a,b) = 1$ we can factor

$$a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

and $b = q_1^{f_1} q_2^{f_2} \cdots q_\ell^{f_\ell}$

where $p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_\ell$ are all distinct primes.

So,

$$\begin{aligned}
\sigma(ab) &= \sigma\left(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} q_1^{f_1} q_2^{f_2} \cdots q_\ell^{f_\ell}\right) \\
&\overset{(2)}{=} \sigma(p_1^{e_1})\sigma(p_2^{e_2}) \cdots \sigma(p_k^{e_k})\sigma(q_1^{f_1})\sigma(q_2^{f_2}) \cdots \sigma(q_\ell^{f_\ell}) \\
&\overset{(2)}{=} \sigma\left(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}\right) \sigma\left(q_1^{f_1} q_2^{f_2} \cdots q_\ell^{f_\ell}\right) \\
&= \sigma(a)\sigma(b).
\end{aligned}$$

Ex 36

$a = 4$

$b = 6$

$\gcd(4,6) = 2 \neq 1$

$\sigma(ab) = \sigma(24) = 1+2+3+4+6+8+12+24 = 60$

$\sigma(a)\sigma(b) = \sigma(4)\sigma(6) = (1+2+4)(1+2+3+6)$

$= (7)(12) = 84$

$\gcd(4,6) = 2 \neq 1$

$\sigma(4 \cdot 6)$

$\neq \sigma(4)\sigma(6)$

Thm 37 (Euclid - Thm 29 redone)

Let $n \geq 2$.

If $2^n - 1$ is prime,

then $2^{n-1}(2^n - 1)$ is perfect.

proof: Note that $2^n - 1$ is an odd prime.

Let $x = 2^{n-1} \cdot p$ where $p = 2^n - 1$. Then $\sigma(x) = \sigma(2^{n-1} p) = \sigma(2^{n-1})\sigma(p)$

$\boxed{\text{p is odd so } p \neq 2}$

$(2^{n-1})\,\sigma(p)$

$$\left(\frac{2^{(n-1)+1}-1}{2-1}\right)\left(\frac{p^2-1}{p-1}\right)$$

$$= \underbrace{(2^n-1)}_{p}\left[\frac{(p-1)(p+1)}{p-1}\right]$$

$$= p(p+1)$$

$$= p(2^n-1+1) = p\cdot 2^n = 2\cdot 2^{n-1}\cdot p$$

$$= 2X.$$

Since $\sigma(X) = 2X$ we have that X is perfect. $\blacksquare$

odd

$=2$