No Class on Weds

Use it as a paper writing / presentation making day

3/2
Mon
week 7

Last time, we saw that if n is an odd perfect number then $n \equiv 1 \pmod 4$.

That is, there are no odd perfect numbers that are $3 \pmod 4$.

In 1953, Touchard showed that if n is an odd perfect number, then $n \equiv 1 \pmod{12}$ or $n \equiv 9 \pmod{36}$.

Touchard used this formula:

$$\frac{n^2(n-1)}{12}\sigma(n) = \sum_{k=1}^{n-1}\left[5k(n-k)-n^2\right]\sigma(k)\sigma(n-k)$$

In 2002, Judy Holdener proved the same result in a simpler way. We will go over Holdener's paper today.

Ex 68: $n = 35 = 6(6) - 1 = 6k - 1$

$\sqrt{35} \approx 5.916\ldots$

divisors of n: $1, \; 5 \; \vdots \; 7, \; 35$

factors come in "pairs" since 35 is not a perfect square.

$$\sigma(n) = \sigma(35) = 1 + 5 + 7 + 35$$

regroup factors by "pairs"

$$= (1 + 35) + (5 + 7)$$

$$= \underbrace{\left(1 + \frac{35}{1}\right) + \left(5 + \frac{35}{5}\right)}_{\text{This is how we will group pairs in lemma 69}}$$

see next page

$$\equiv (1-1)+(-1+1) \pmod 3$$

$35 \equiv -1 \pmod 3$

$5 \equiv -1 \pmod 3$

$7 \equiv 1 \pmod 3$

$$\equiv 0 \pmod 3.$$

Lemma 69: If n is an odd number and $n = 6k-1$ where $k \geq 1$, then n is not perfect.

Proof: Let n be an odd integer with $n = 6k-1$ where $k \geq 1$.

Then,

$$\begin{aligned} n = 6k-1 &\equiv 3(2k)-1 \pmod 3 \\ &\equiv 0-1 \pmod 3 \\ &\equiv -1 \pmod 3. \end{aligned}$$

Let's show n is not a perfect square.

Suppose $n = x^2$ where $x \in \mathbb{Z}$.

Then, $x \equiv 0 \pmod 3$ or $x \equiv 1 \pmod 3$, or $x \equiv 2 \pmod 3$.

If $x \equiv 0 \pmod 3$, then $n = x^2 \equiv 0^2 \pmod 3 \equiv 0 \pmod 3 \not\equiv -1 \pmod 3$

If $x \equiv 1 \pmod 3$, then $n = x^2 = 1^2 \pmod 3 \equiv 1 \pmod 3 \not\equiv -1 \pmod 3$

If $x \equiv 2 \pmod 3$, then $n = x^2 = 2^2 \pmod 3 \equiv 4 \pmod 3 \equiv 1 \pmod 3 \not\equiv -1 \pmod 3$.

In all cases, $n \not\equiv -1 \pmod 3$.

So, n cannot be a perfect square.

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$-1 \pmod 3$.

So, the divisors of n come in pairs.

So, for any divisor d of n,

$$d \cdot \frac{n}{d} = n \equiv -1 \pmod 3.$$

divisor d and its "pair" $\frac{n}{d}$

This implies that either

$$d \equiv 1 \pmod 3 \text{ and } \frac{n}{d} \equiv -1 \pmod 3$$

OR

$$d \equiv -1 \pmod 3 \text{ and } \frac{n}{d} \equiv 1 \pmod 3.$$

Either way

$$\left(d + \frac{n}{d}\right) \equiv 0 \pmod 3.$$

So,

$$\sigma(n) = \sum_{d \mid n} d$$

$$= \sum_{\substack{d \mid n \\ 1 \le d < \sqrt{n}}} \left(d + \frac{n}{d} \right)$$

$$\equiv \sum_{\substack{d \mid n \\ 1 \le d < \sqrt{n}}} 0 \pmod{3} \equiv 0 \pmod{3}.$$

Why can't n be a perfect number?

If it was then,

$$\sigma(n) = 2n.$$

Then, $\sigma(n) = 2n \equiv 2(-1) \pmod 3$

$$\equiv -2 \pmod 3$$

$$\equiv 1 \pmod 3.$$

But then $\sigma(n) \equiv 0 \pmod{3}$

and $\sigma(n) \equiv 1 \pmod{3}$

which can't happen.

So, n can't be perfect. $\blacksquare$

Theorem 70 (Touchard 1953, Holdener 2002)

If n is an odd perfect number, then

$$\underbrace{n \equiv 1 \pmod{12}}_{n=12k+1} \quad \text{or} \quad \underbrace{n \equiv 9 \pmod{36}}_{n=36k+9}.$$

proof: Let n be an odd perfect number.

By Euler, $n \equiv 1 \pmod{4}$.

Dividing 6 into n gives that $n = 6k$, $n = 6k+1$, $n = 6k+2$, $n = 6k+3$, $n = 6k+4$, or $n = 6k+5$.

Since n is odd,

$n = 6k+1$, $n = 6k+3$, or $n = 6k+5$.

Since $6k+5 \equiv 5 \pmod 6 \equiv -1 \pmod 6$

by Lemma 69, $n \neq 6k+5$.

So either $n = 6k+1$

or $n = 6k+3$.

So, either

$n \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 6$

OR $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 6$.

Let's consider n modulo 12.

By sidework $\longrightarrow$

either $n \equiv 1 \pmod{12}$ or $n \equiv 9 \pmod{12}$

sidework

✗ $0 + 12\ell \equiv 0 + 4(3\ell) \equiv 0 \pmod 4$

$1 + 12\ell \equiv 1 \pmod 4$ and $1 + 12\ell \equiv 1 \pmod 6$

✗ $2 + 12\ell \equiv 2 \pmod 4$

✗ $3 + 12\ell \equiv 3 \pmod 4$

✗ $4 + 12\ell \equiv 0 \pmod 4$

✗ $5 + 12\ell \equiv 1 \pmod 4$ and $5 \pmod 6$

✗ $6 + 12\ell \equiv 2 + 4 + 4(3\ell) \equiv 2 \pmod 4$

✗ $7 + 12\ell \equiv 3 \pmod 4$

✗ $8 + 12\ell \equiv 0 \pmod 4$

$9 + 12\ell \equiv 1 \pmod 4$, $9 + 12\ell \equiv 3 \pmod 6$

✗ $10 + 12\ell \equiv 2 \pmod 4$

✗ $11 + 12\ell \equiv 3 \pmod 4$

If $n \equiv 9 \pmod{12}$,

then $n = 12m + 9$

where $m \in \mathbb{Z}$, $m \geq 0$.

Let's show $3 \mid m$.

Suppose $3 \nmid m$.

Then,

$$\sigma(n) = \sigma(12m+9)$$
$$= \sigma(3\cdot(4m+3))$$
$$= \sigma(3)\,\sigma(4m+3)$$

$\gcd(3, 4m+3) = 1$

Since $3 \nmid m$,
we get $3 \nmid (4m+3)$
Why? If $4m+3 = 3z$
then $3[z-1] = 4m$.

Then $3 \mid 4m$.
By thm 38, since $\gcd(3,4) = 1$, $3 \mid m$.
which can't happen.

$$(1+3)\,\sigma(4m+3) = 4\,\sigma(4m+3)$$

Then,

$$\sigma(n) = 4\,\sigma(4m+3) \equiv \boxed{0 \pmod 4}.$$

Since n is perfect,

$$\begin{aligned}\sigma(n) = 2n = 2(12m+9) &= 24m+18 \\ &\equiv (0+2) \bmod 4 \\ &\equiv \boxed{2 \pmod 4}.\end{aligned}$$

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So, $3 \nmid m$ leads to a contradiction.

So, $3 \mid m$.

Thus, $m = 3q$ where $q \in \mathbb{Z}$.

So, $n = 12m + 9 = 36q + 9$.

Thus, if $n \equiv 9 \pmod{12}$,
then $n \equiv 9 \pmod{36}$.

So, either

$$n \equiv 1 \pmod{12}$$

or

$$n \equiv 9 \pmod{36}.$$

n is an odd perfect number

$\boxed{n \equiv 1 \pmod 4}$

- $0 \pmod{12}$
- $\boxed{1 \pmod{12}}$
- $2 \pmod{12}$
- $3 \pmod{12}$
- $4 \pmod{12}$
- $5 \pmod{12}$
- $6 \pmod{12}$
- $7 \pmod{12}$
- $8 \pmod{12}$
- $\boxed{9 \pmod{12}}$
  - $\boxed{9 \pmod{36}}$
  - $21 \pmod{36}$
  - $33 \pmod{36}$
- $10 \pmod{12}$
- $11 \pmod{12}$

Tim Roberts shows.

If n is an odd perfect number then either
$n \equiv 1 \pmod{12}$ or
$n \equiv 117 \pmod{468}$ or
$n \equiv 81 \pmod{324}$.