

No
Class
on
Weds

Use it as
a paper
writing/presentation
making day

3/2
Mon
week 7

Last time, we saw that
if n is an odd perfect
number then $n \equiv 1 \pmod{4}$.

That is, there are no odd
perfect numbers that
are $3 \pmod{4}$.

In 1953, Touchard showed that if n is an odd perfect number, then $n \equiv 1 \pmod{12}$

or $n \equiv 9 \pmod{36}$.

Touchard used this formula:

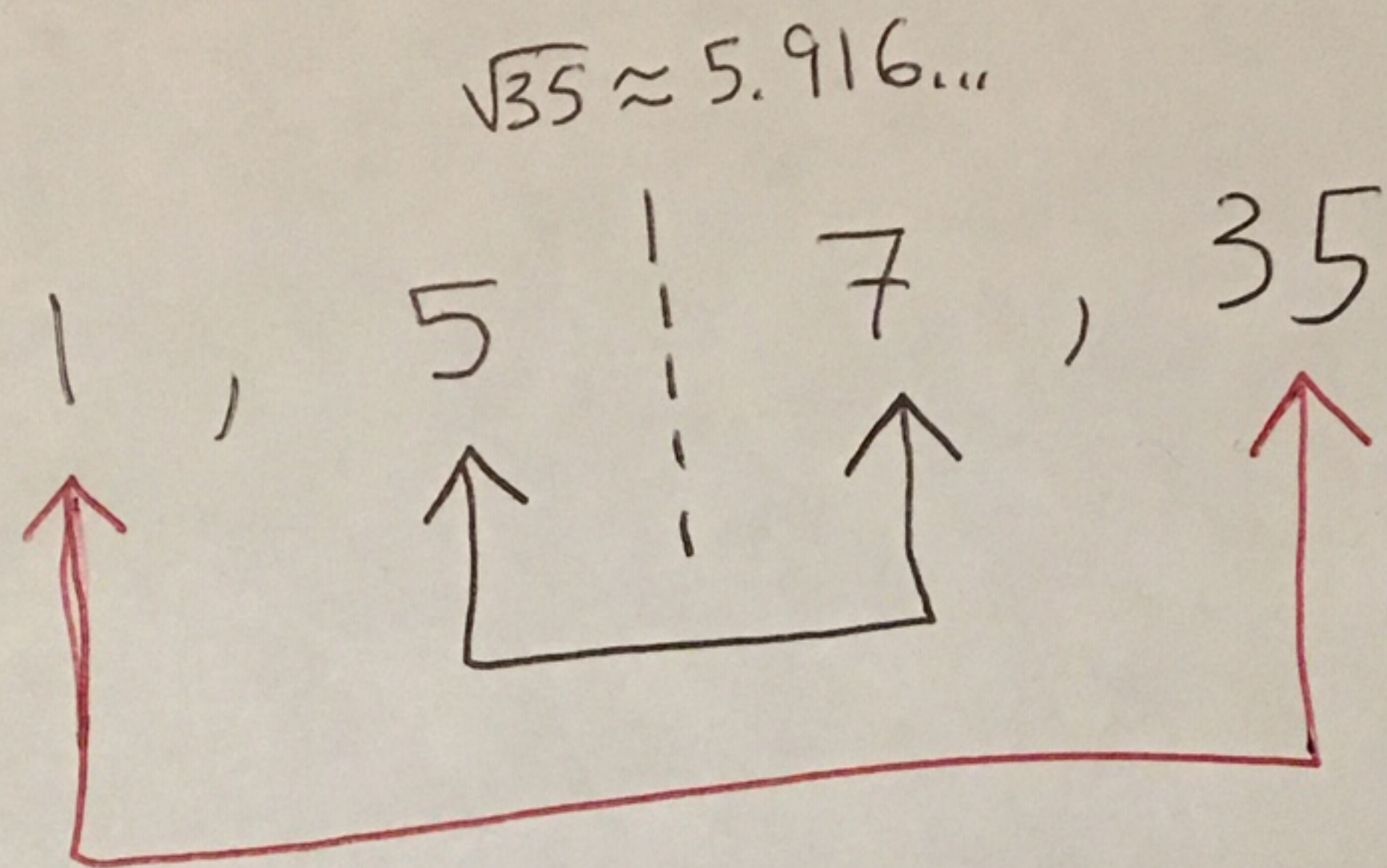
$$\frac{n^2(n-1)}{12} \sigma(n) = \sum_{k=1}^{n-1} [5k(n-k) - n^2] \sigma(k) \sigma(n-k)$$

In 2002,

Judy Holdener proved the same result in a simpler way. We will go over Holdener's paper today.

Ex 68: $n = 35 = 6(6) - 1 = 6k - 1$

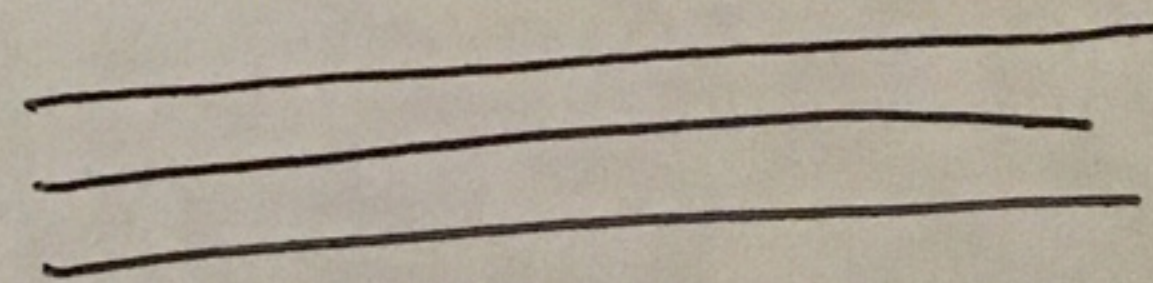
divisors of n :



factors
come in
"pairs" since
35 is not
a perfect
square.

$$\begin{aligned}\sigma(n) &= \sigma(35) = 1 + 5 + 7 + 35 \\ &= (1 + 35) + (5 + 7) \\ &= \left(1 + \frac{35}{1}\right) + \left(5 + \frac{35}{5}\right)\end{aligned}$$

regroup
factors
by "pairs"



↑
see
next
page

This is how we will
group pairs in lemma 69

$$(1-1) + (-1+1) \pmod{3}$$

$$\begin{aligned} 35 &\equiv -1 \pmod{3} \\ 5 &\equiv -1 \pmod{3} \\ 7 &\equiv 1 \pmod{3} \end{aligned} \quad \equiv 0 \pmod{3}$$

Lemma 69: If n is an odd number and $n=6k-1$ where $k \geq 1$, then n is not perfect.

Proof: Let n be an odd integer with $n=6k-1$ where $k \geq 1$.

Then,

$$\begin{aligned} n=6k-1 &\equiv 3(2k)-1 \pmod{3} \\ &\equiv 0-1 \pmod{3} \\ &\equiv -1 \pmod{3}. \end{aligned}$$

Let's show n is not a perfect square.

Suppose $n = x^2$ where $x \in \mathbb{Z}$.

Then, $x \equiv 0 \pmod{3}$ or $x \equiv 1 \pmod{3}$, or $x \equiv 2 \pmod{3}$.

If $x \equiv 0 \pmod{3}$, then $n = x^2 \equiv 0^2 \pmod{3} \equiv 0 \pmod{3} \not\equiv -1 \pmod{3}$

If $x \equiv 1 \pmod{3}$, then $n = x^2 \equiv 1^2 \pmod{3} \equiv 1 \pmod{3} \not\equiv -1 \pmod{3}$

If $x \equiv 2 \pmod{3}$, then $n = x^2 \equiv 2^2 \pmod{3} \equiv 4 \pmod{3} \equiv 1 \pmod{3} \not\equiv -1 \pmod{3}$.

In all cases, $n \not\equiv -1 \pmod{3}$.

So, n cannot be a perfect square.

So, the divisors of n
come in pairs.

So, for any divisor d
of n ,

$$d \cdot \frac{n}{d} = n \equiv -1 \pmod{3}$$

divisor d and
its "pair" $\frac{n}{d}$

This implies that either
 $d \equiv 1 \pmod{3}$ and $\frac{n}{d} \equiv -1 \pmod{3}$

OR

$d \equiv -1 \pmod{3}$ and $\frac{n}{d} \equiv 1 \pmod{3}$.

Either way

$$\left(d + \frac{n}{d}\right) \equiv 0 \pmod{3}$$

So,

$$\sigma(n) = \sum_{d|n} d$$

$$= \sum_{\substack{d|n \\ 1 \leq d < \sqrt{n}}} \left(d + \frac{n}{d} \right)$$

$$\equiv \sum_{\substack{d|n \\ 1 \leq d < \sqrt{n}}} 0 \pmod{3} \equiv 0 \pmod{3}.$$

Why can't n be a perfect number?

If it was then,

$$\sigma(n) = 2n.$$

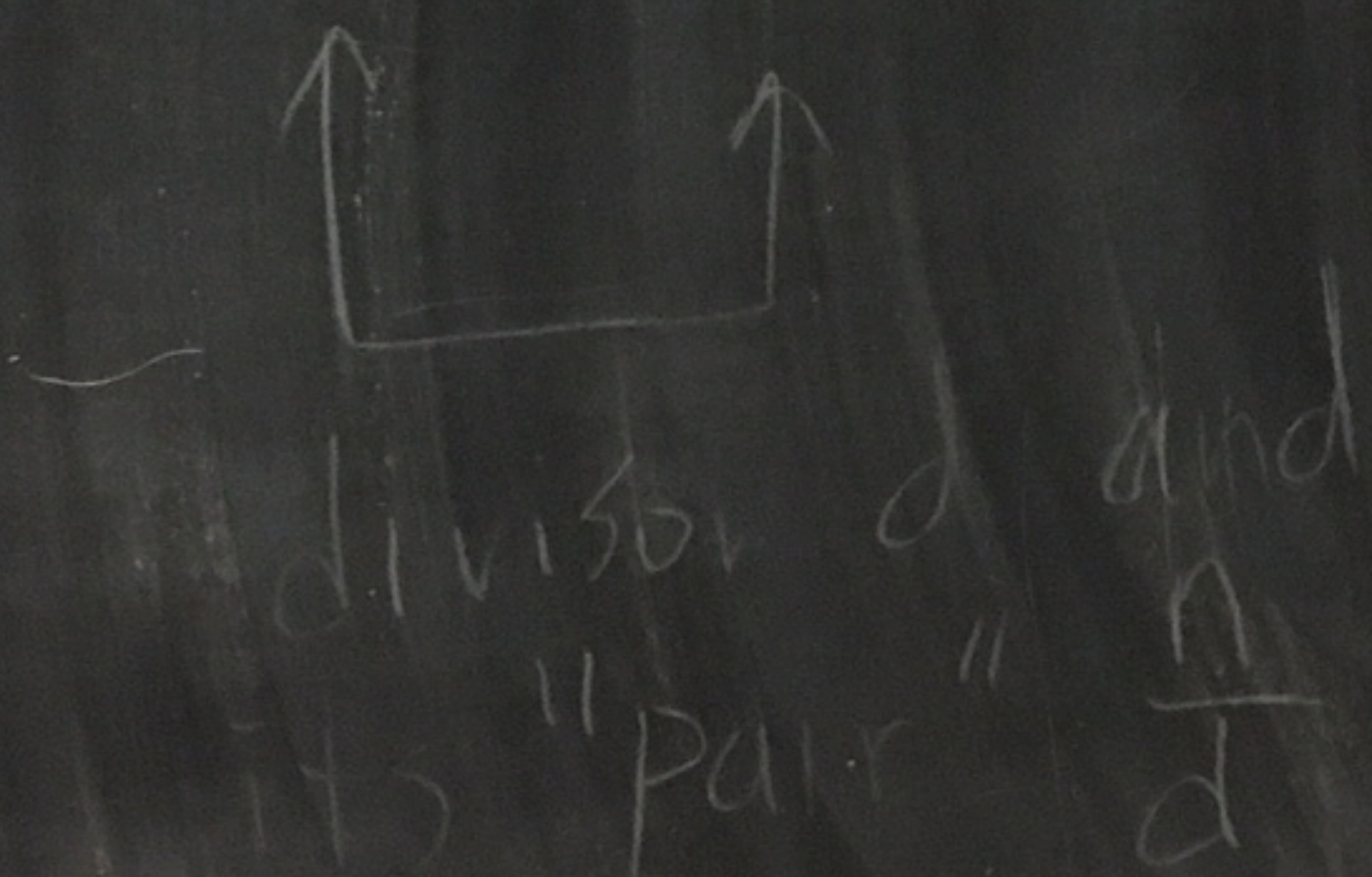
Then, $\sigma(n) = 2n \equiv 2(-1) \pmod{3}$

$$\equiv -2 \pmod{3}$$

$$\equiv 1 \pmod{3}.$$

But then $\sigma(n) \equiv 0 \pmod{3}$
and $\sigma(n) \equiv 1 \pmod{3}$
which can't happen.

So, n can't be perfect. \square



Theorem 70 (Touchard 1953,
Holdener 2002)

If n is an odd perfect
number, then

$$n \equiv 1 \pmod{12}$$

or $n \equiv 9 \pmod{36}$.

$$n = 12k + 1$$

$$n = 36k + 9$$

proof: Let n be an
odd perfect number.

By Euler, $n \equiv 1 \pmod{4}$.

Dividing 6 into n gives that

$$n = 6k, n = 6k + 1, n = 6k + 2,$$

$$n = 6k + 3, n = 6k + 4, \text{ or } n = 6k + 5.$$

Since n is odd,

$n = 6k+1$, $n = 6k+3$, or $n = 6k+5$.

Since $6k+5 \equiv 5 \pmod{6} \equiv -1 \pmod{6}$
by Lemma 69, $n \neq 6k+5$.

So either $n = 6k+1$

or $n = 6k+3$.

So, either

$$n \equiv 1 \pmod{4} \text{ and } n \equiv 1 \pmod{6}$$

OR $n \equiv 1 \pmod{4} \text{ and } n \equiv 3 \pmod{6}$.

Let's consider n modulo 12.

By sidework \longrightarrow

either $n \equiv 1 \pmod{12}$ or $n \equiv 9 \pmod{12}$

sidework

$$\times 0 + 12l \equiv 0 + 4(3l) \equiv 0 \pmod{4}$$

$$\circled{1 + 12l \equiv 1 \pmod{4} \text{ and } 1 + 12l \equiv 1 \pmod{6}}$$

$$\times 2 + 12l \equiv 2 \pmod{4}$$

$$\times 3 + 12l \equiv 3 \pmod{4}$$

$$\times 4 + 12l \equiv 0 \pmod{4}$$

$$\times 5 + 12l \equiv 1 \pmod{4} \text{ and } 5 \pmod{6}$$

$$\times 6 + 12l \equiv 2 + 4 + 4(3l) \equiv 2 \pmod{4}$$

$$\times 7 + 12l \equiv 3 \pmod{4}$$

$$\times 8 + 12l \equiv 0 \pmod{4}$$

$$\circled{9 + 12l \equiv 1 \pmod{4}, 9 + 12l \equiv 3 \pmod{6}}$$

$$\times 10 + 12l \equiv 2 \pmod{4}$$

$$\times 11 + 12l \equiv 3 \pmod{4}$$

If $n \equiv 9 \pmod{12}$,

then $n = 12m + 9$

where $m \in \mathbb{Z}$, $m \geq 0$.

Let's show $3 \mid m$.

Suppose $3 \nmid m$.

Then,

$$\sigma(n) = \sigma(12m + 9)$$

$$= \sigma(3 \cdot (4m + 3))$$

$$\underline{\underline{\sigma(3) \sigma(4m + 3)}}$$

$$\text{gcd}(3, 4m + 3) = 1$$

Since $3 \nmid m$,
we get $3 \nmid (4m + 3)$
why? If $4m + 3 = 3z$
then $3[z - 1] = 4m$.

Then $3 \mid 4m$.

By thm 38, since
 $\text{gcd}(3, 4) = 1$, $3 \mid m$,
which can't happen.

$$(1+3) \sigma(4m+3) = 4 \sigma(4m+3)$$

Then,

$$\sigma(n) = 4 \sigma(4m+3) \equiv 0 \pmod{4}.$$

Since n is perfect,

$$\sigma(n) = 2n = 2(12m+9) = 24m+18$$

$$\equiv (0+2) \pmod{4}$$

$$\equiv 2 \pmod{4}.$$

3)

So, $3 \nmid m$ leads to a contradiction.

So, $3 \mid m$.

Thus, $m = 3q$ where $q \in \mathbb{Z}$.

So, $n = 12m + 9 = 36q + 9$.

Thus, if $n \equiv 9 \pmod{12}$,
then $n \equiv 9 \pmod{36}$.

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) mod 4

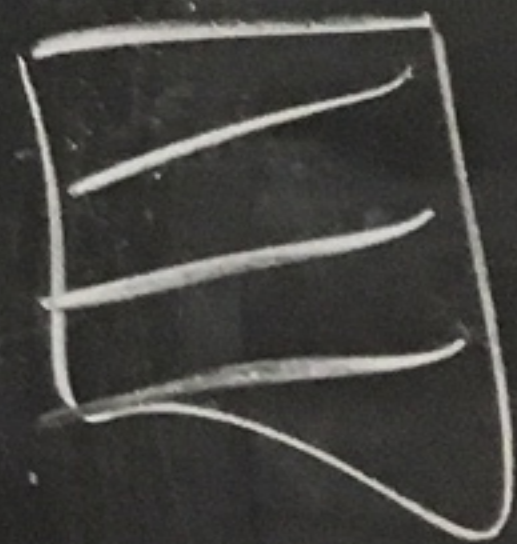
4).

So, either

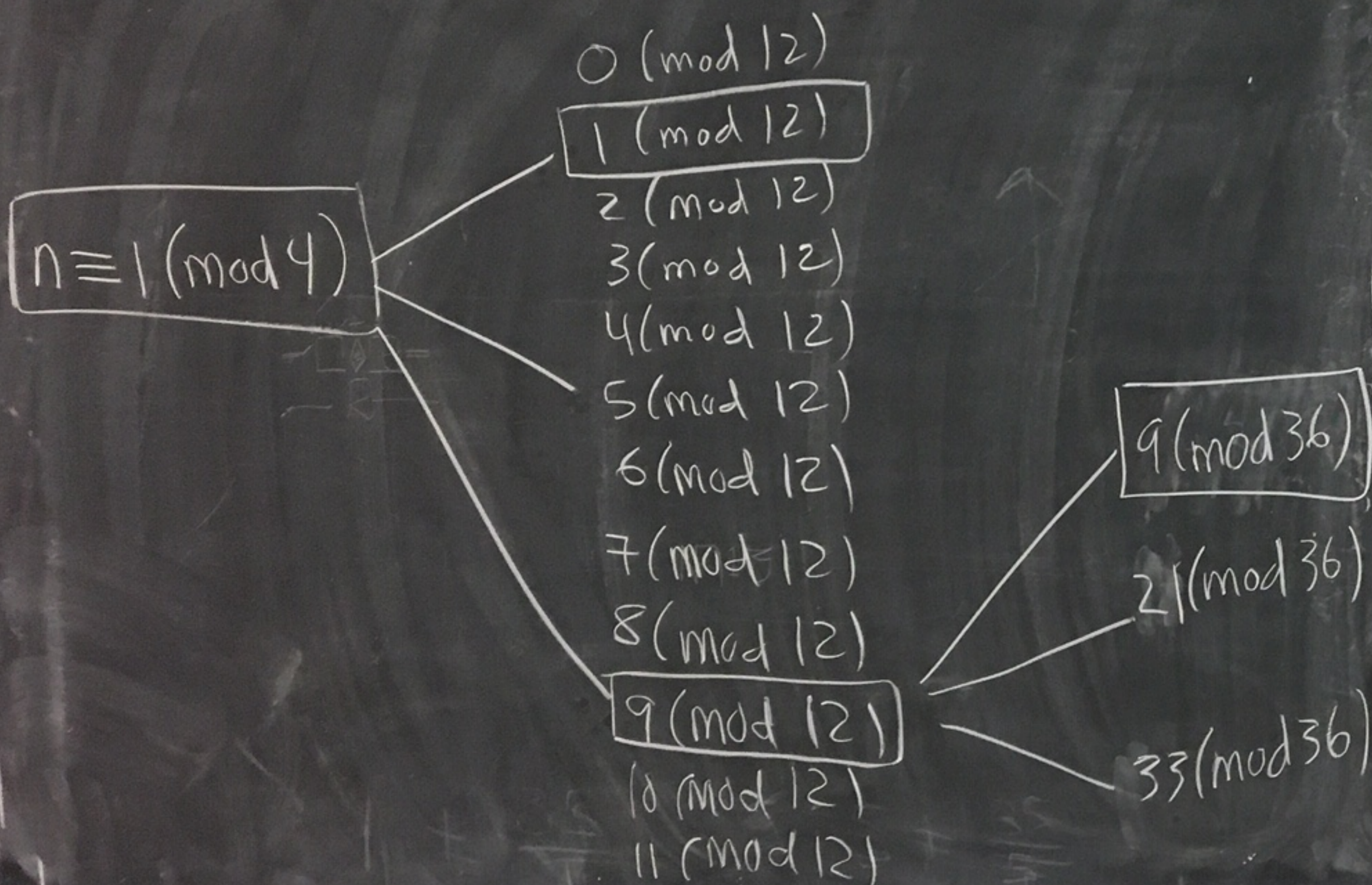
$$n \equiv 1 \pmod{12}$$

or

$$n \equiv 9 \pmod{36}$$



n is an odd perfect number



Tim Roberts shows

If n is an odd perfect number then either

$n \equiv 1 \pmod{12}$ or
 $n \equiv 117 \pmod{468}$ or
 $n \equiv 81 \pmod{324}$.