Tuesday
1/28

From our table

$\sqrt{a^2-x^2}$
$x = a\sin(\theta)$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$1-\sin^2(\theta) = \cos^2(\theta)$

7.4 continued

Ex: Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$$\int \frac{\sqrt{9-(3\sin(\theta))^2}}{(3\sin(\theta))^2}\,3\cos(\theta)\,d\theta$$

$$x = 3\sin(\theta)$$
$$dx = 3\cos(\theta)\,d\theta$$

$$= 3\int \frac{\sqrt{9-9\sin^2(\theta)}}{9\sin^2(\theta)}\cos(\theta)\,d\theta$$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

$\frac{3}{9}$

$\sqrt{ab}=\sqrt{a}\sqrt{b}$

$$\frac{3}{9}\int \frac{\sqrt{9}\sqrt{1-\sin^2(\theta)}}{\sin^2(\theta)}\cos(\theta)\,d\theta = \int \frac{\sqrt{\cos^2(\theta)}}{\sin^2(\theta)}\cos(\theta)\,d\theta$$

$$= \int \frac{\cos(\theta)}{\sin^2(\theta)}\cos(\theta)\,d\theta = \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta = \int \cot^2(\theta)\,d\theta$$

$$\sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

$$= \cos(\theta)$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$= \int (\csc^2(\theta)-1)\,d\theta = -\cot(\theta) - \theta + C$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1+\cot^2(\theta) = \csc^2(\theta)$$

$\theta)\,d\theta$

$s(\theta)\,d\theta$

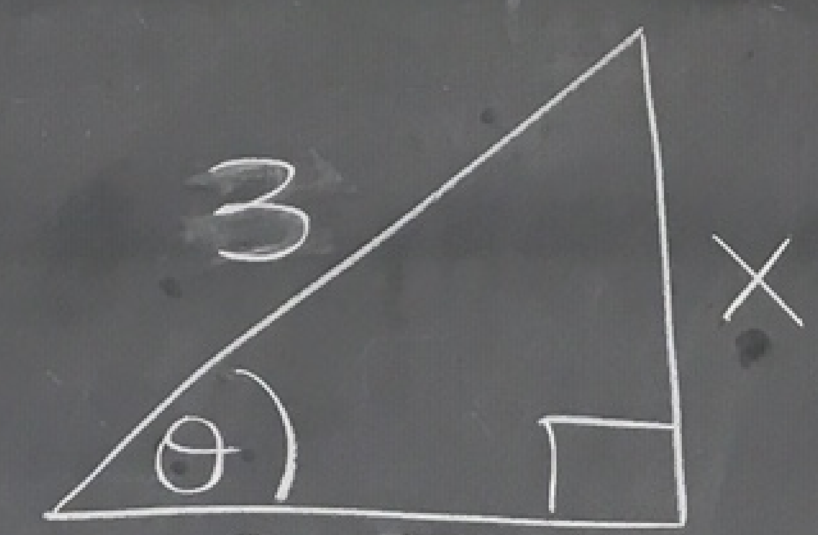
Go back to original substitution

$$x = 3\sin(\theta)$$

$$\frac{x}{3} = \sin(\theta)$$

$$\frac{\text{opp}}{\text{hyp}} = \frac{x}{3} = \sin(\theta)$$

$$\boxed{\sin^{-1}\left(\frac{x}{3}\right) = \theta}$$



$$? = \sqrt{9-x^2}$$

$$\boxed{\begin{aligned} ?^2 + x^2 &= 3^2 \\ ? &= \sqrt{9-x^2} \end{aligned}}$$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

$$= -\cot(\theta) - \theta + C$$

$$= -\frac{\text{adj}}{\text{opp}} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

Ex: Evaluate $\displaystyle\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

$$= \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(\sqrt{4\left(x^2+\frac{9}{4}\right)}\right)^3}\,dx = \frac{1}{8}\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(\sqrt{x^2+\left(\frac{3}{2}\right)^2}\right)^3}\,dx$$

Sub to make:

$x = \frac{3}{2}\tan(\theta)$

$dx = \frac{3}{2}\sec^2(\theta)\,d\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

When $x = \frac{3\sqrt{3}}{2}$, we get $\frac{3\sqrt{3}}{2} = \frac{3}{2}\tan(\theta)$

or $\tan(\theta) = \sqrt{3}$ or $\theta = \frac{\pi}{3}$

$\left[\text{At } \theta = \frac{\pi}{3},\ \tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}\right]$

When $x = 0$, we get $0 = \frac{3}{2}\tan(\theta)$ or $\tan(\theta) = 0$

or $\theta = 0$

Ergo,

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

$$= \frac{1}{8}\int_0^{\pi/3} \frac{\left(\frac{3}{2}\tan(\theta)\right)^3}{\left(\sqrt{\left(\frac{3}{2}\tan(\theta)\right)^2+\frac{9}{4}}\right)^3} \cdot \frac{3}{2}\sec^2(\theta)\,d\theta$$

$$\frac{3}{16}\int_0^{\pi/3} \frac{\frac{27}{8}\tan^3(\theta)}{\left(\sqrt{\frac{9}{4}\left(\tan^2(\theta)+1\right)}\right)^3}\sec^2(\theta)\,d\theta$$

$$= \frac{3}{16}\cdot\frac{\left(\frac{27}{8}\right)}{\left(\frac{27}{8}\right)}\int_0^{\pi/3} \frac{\tan^3(\theta)\sec^2(\theta)\,d\theta}{\left(\sqrt{\tan^2(\theta)+1}\right)^3}$$

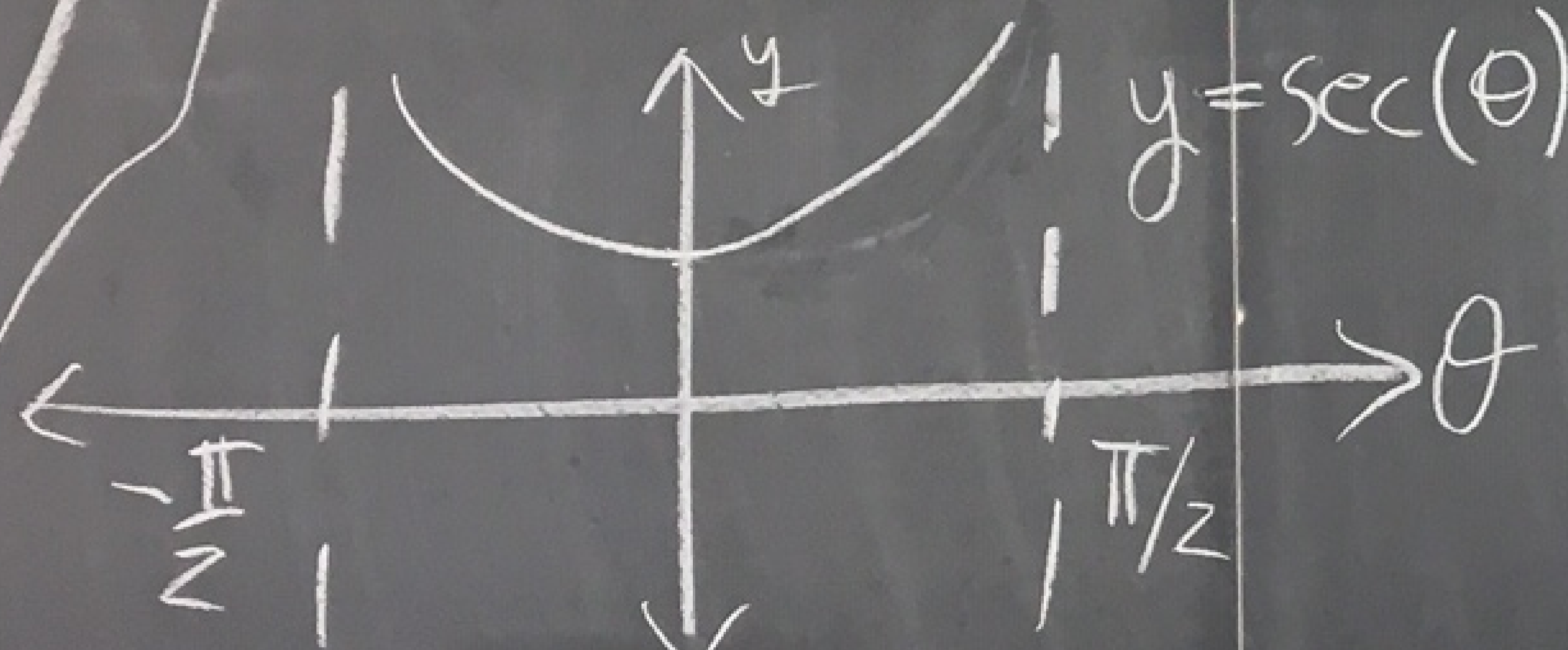
$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3(\theta)\sec^2(\theta)}{\left(\sqrt{\sec^2(\theta)}\right)^3}\,d\theta$$

$)d\theta$

θ

$$\frac{3}{16}\int_0^{\pi/3} \frac{\tan^3(\theta)\sec^2(\theta)}{\sec^3(\theta)}\,d\theta$$

When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



So, $\sqrt{\sec^2(\theta)} = |\sec(\theta)| = \sec(\theta)$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3(\theta)}{\sec(\theta)}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\left(\frac{\sin^3(\theta)}{\cos^3(\theta)}\right)}{\left(\frac{1}{\cos(\theta)}\right)}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3(\theta)}{\cos^2(\theta)}\,d\theta$$