Weds
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8.2
continued

Ex: Does $\left\{\frac{\cos(n)}{n^2+1}\right\}_{n=1}^{\infty}$ converge or diverge?

$$-1 \leq \cos(n) \leq 1$$

$$-\frac{1}{n^2+1} \leq \frac{\cos(n)}{n^2+1} \leq \frac{1}{n^2+1}$$

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

$$\boxed{n^2+1 \to \infty}$$

Squeeze thm

If $a_n \le b_n \le c_n$

and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

then $\lim_{n\to\infty} b_n = L$

OR

$$\lim_{n\to\infty} \frac{1}{n^2+1} = \lim_{n\to\infty} \frac{1/n^2}{1+1/n^2} = \frac{0}{1+0} = 0$$

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

$$\lim_{n\to\infty} \frac{-1}{n^2+1} = 0$$

$$-\frac{1}{n^2+1} \le \frac{\cos(n)}{n^2+1} \le \frac{1}{n^2+1}$$

$\downarrow$ as $n\to\infty$ $\downarrow$ as $n\to\infty$

0 0

So, by the squeeze thm

$$\lim_{n\to\infty} \frac{\cos(n)}{n^2+1} = 0$$

Thm: If $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$

$\frac{(-1)^2}{2^2} = \frac{1}{4}$

Ex: Does $\left\{ \frac{(-1)^n}{n^2} \right\}$ converge or diverge?

$a_n = \frac{(-1)^n}{n^2}$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, \ldots$

$a_1, a_2, a_3, a_4, \ldots$

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n^2}\right| = \lim_{n\to\infty}\frac{|(-1)^n|}{|n^2|} = \lim_{n\to\infty}\frac{1}{n^2} = 0$$

$(-1)^n = \pm 1$

$|(-1)^n| = 1$

$n^2 \geq 0$

$|n^2| = n^2$

So, by the thm

$$\lim_{n\to\infty}\frac{(-1)^n}{n^2} = 0.$$

Ex: Does $\left\{ \dfrac{2n+1}{\sqrt{81n^2-3n}} \right\}_{n=1}^{\infty}$ converge or diverge?

Spider sense:

$$\frac{2n+1}{\sqrt{81n^2-3n}} \approx \frac{2n}{\sqrt{81n^2}} = \frac{2n}{9n} = \frac{2}{9}$$

when n is really big

$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{81n^2-3n}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{\frac{1}{n^2}}\sqrt{81n^2-3n}}$$

$\frac{1}{n} = \sqrt{\frac{1}{n^2}}$ ok since rge or diverge ?

$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{81n^2-3n}} = \lim_{n\to\infty} \frac{\frac{1}{n}(2n+1)}{\frac{1}{n}\sqrt{81n^2-3n}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{\frac{1}{n^2}}\sqrt{81n^2-3n}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{81-\frac{3}{n}}}$$

$\frac{1}{n} = \sqrt{\frac{1}{n^2}}$ ok since $n>0$

$$= \frac{\lim_{n\to\infty}(2+\frac{1}{n})}{\lim_{n\to\infty}\sqrt{81-\frac{3}{n}}} = \frac{\lim_{n\to\infty}(2+\frac{1}{n})}{\sqrt{\lim_{n\to\infty}(81-\frac{3}{n})}}$$

Here $f(x)=\sqrt{x}$

$$= \frac{2+0}{\sqrt{81-0}}$$

$$= \frac{2}{9}$$

If $f(x)$ is a continuous function then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

Ex: Does $a_n = \left(\frac{n+5}{n}\right)^n$ converge or diverge?

To deal with the n in the exponent we take ln of the sequence.

$$\lim_{n\to\infty} \ln\left[\left(\frac{n+5}{n}\right)^n\right] = \lim_{n\to\infty} n\cdot\ln\left(\frac{n+5}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{5}{n}\right)}{1/n}$$

$$\lim_{n\to\infty} n = \infty$$

$$\lim_{n\to\infty} \ln\left(\frac{n+5}{n}\right) = \ln\left(\lim_{n\to\infty} \frac{n+5}{n}\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{5}{n}\right)\right)$$

$$= \ln(1+0) = 0$$

$\infty \cdot 0$ situation

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{5}{n}\right)}\cdot\left(-\frac{5}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5}{1+\frac{5}{n}}$$

$$= \frac{5}{1+0} = 5$$

"$\frac{0}{0}$"

$$\frac{5}{n} = 5n^{-1}$$

So, $\lim_{n\to\infty}\left(\frac{n+5}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left[\left(\frac{n+5}{n}\right)^n\right]}$

$$= e^{\lim_{n\to\infty} \ln\left[\left(\frac{n+5}{n}\right)^n\right]} = e^5$$