8.3 - Infinite Series

Thurs 2/13

We want to make sense of an infinite sum:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

Given such a sum we define the <u>partial sums</u>:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots \qquad \vdots$$

verges.

In general,

$$\boxed{\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ S_n &= \sum_{k=1}^{n} a_k \end{aligned}}$$

If $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L$

Where L is a number

then we say that $\sum_{k=1}^{\infty} a_k$ converges to L. If $\lim_{n \to \infty} S_n$ doesn't exist, then we say that $\sum_{k=1}^{\infty} a_k$ diverges

8.

We

of

$\sum_{k}^{\infty}$

Ex: $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$

$\zeta(s) \leftarrow$ Riemann zeta function

$\underbrace{2, 3, 5, 7, 11, 13, 17,}_{\text{7 primes } \leq 18}$ $19, 23, \ldots$

$x = 18$

verges.

Let's estimate $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$

$S_1 = \frac{1}{1^2} = \boxed{1}$

$S_2 = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4} = \boxed{1.25}$

$S_3 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36} \approx \boxed{1.3611111\ldots}$

$S_4 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$

$= \frac{205}{144} \approx \boxed{1.42361111\ldots}$

$S_5 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$

$= \frac{5269}{3600} \approx \boxed{1.4636111\ldots}$

$\vdots$

$$S_{100} = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{99^2} + \frac{1}{100^2} \approx 1.63498390018\dots$$

$$S_{1000} = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{999^2} + \frac{1}{1000^2} \approx 1.64393456668\dots$$

$$S_{100,000} \approx 1.644924066898\dots$$

People have figured out that $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$

$$= \lim_{n \to \infty} S_n = \frac{\pi^2}{6} \approx 1.644934066848\dots$$

Ex: Consider the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

Does it converge or diverge?

Let's see some partial sums.

$$S_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{4}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{9}{12} = \frac{3}{4}$$

We guess that

$$S_n = \frac{n}{n+1} \longrightarrow 1$$

How can we be sure?
Use partial fractions.

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

multiply both sides by $k(k+1)$

Solve for A and B.

$$\boxed{1 = A(k+1) + Bk}$$

Solve for A

Plug in $k=0$

$1 = A(1) + B(0)$

$\boxed{A = 1}$

Solve for B

Plug in $k=-1$

$1 = A(0) + B(-1)$

$\boxed{B = -1}$

So, $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left[\frac{A}{k} + \frac{B}{k+1} \right]$$

$$= \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1} \right]$$

Let's see some partial sums now

$$S_1 = \left(\frac{1}{1} - \frac{1}{2} \right) = 1 - \frac{1}{2}$$

$$S_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3} \right) = 1 - \frac{1}{3}$$

$$S_3 = \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4} \right)$$

$$= 1 - \frac{1}{4}$$

$$S_4 = \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5} \right)$$

$$= 1 - \frac{1}{5}$$

So in general

$$S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

So, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1}$

divide by n on top/bottom

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{1+0} = 1.$$

So, $\boxed{\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \text{ converges to } 1}$